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A SECOND-ORDER SELF-ADJOINT EQUATION ON A TIME SCALE

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ABSTRACT. In this paper we examine the dynamic equation $[p(t)x^{\Delta}(t)]^{\nabla} + q(t)x(t) = 0$ on a time scale. Little work has been done on this equation, which combines both the delta and nabla derivatives. Several preliminary results are established, including Abel's formula and its converse. We then proceed to investigate oscillation and disconjugacy of this dynamic equation.

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1. NABLA DERIVATIVES

In this paper, we are concerned with the second-order self-adjoint dynamic equation

$$[p(t)x^{\Delta}]^{\nabla} + q(t)x = 0.$$

We begin our work by reviewing some properties of the nabla derivative, and then in Section 2, we proceed to establish several results concerning the interaction of the two types of derivatives. In the third section of the paper, we develop Abel's formula, and its converse, which we then use to prove a reduction of order theorem. In the final section, we turn our attention to oscillation and disconjugacy, establishing first an analogue of the Sturm separation theorem, and then, via the Pólya and Trench factorizations, we demonstrate the existence of recessive and dominant solutions of the self-adjoint equation.

Here it is assumed that the reader is already familiar with the basic notions of calculus on a time scale, using the delta-derivative (or Δ-derivative). The reader may be less familiar, however, with the nabla-derivative (or ∇-derivative) on a time scale, as developed by Atıcı and Guseinov [1], and so we include here a brief introduction to its properties, as previously established in other works, stating them without proof. Readers desiring more information are directed to [2] and [1].

Throughout, we assume that $\mathbb{T}$ is a time scale. The notation $[a, b]$ is understood to mean the real interval $[a, b]$ intersected with $\mathbb{T}$.

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Definition 1.1. Let $\mathbb{T}$ be a time scale. For $t > \inf(\mathbb{T})$, the *backward jump operator*, $\rho(t)$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

and the *backward graininess function* $\nu(t)$ is defined by

$$\nu(t) := t - \rho(t).$$

If $f : \mathbb{T} \to \mathbb{R}$, the notation $f^\rho(t)$ is understood to mean $f(\rho(t))$.

Remark 1.2. Here, we retain the original definition of $\nu(t)$. This definition is consistent with the original literature published on ∇-derivatives. It is inconsistent, however with the current work on α-derivatives. When working with α-derivatives, the α-graininess, μ_α is defined to be $\mu_\alpha := \alpha(t) - t$. When $\alpha(t) = \rho(t)$, then, we would have $\mu_\rho := \rho(t) - t = -\nu(t)$. This inconsistency is unfortunate, but we feel it is more important that we remain consistent with the way $\nu(t)$ was defined in previously published work. To minimize confusion, we recommend the notation $\mu_\rho(t) = \rho(t) - t$ be used in work that is to be interpreted in the more general α-derivative setting.

Definition 1.3. Define the set $\mathbb{T}_\kappa$ as follows: If $\mathbb{T}$ has a right-scattered minimum m, set $\mathbb{T}_\kappa := \mathbb{T} \setminus \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$.

Definition 1.4. Let $t \in \mathbb{T}_\kappa$. Then the *∇-derivative of f at t*, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \le \varepsilon|\rho(t) - s|$$

for all $s \in U$.

For $\mathbb{T} = \mathbb{R}$, the ∇-derivative is just the usual derivative. That is, $f^\nabla = f'$. For $\mathbb{T} = \mathbb{Z}$ the ∇-derivative is the backward difference operator, $f^\nabla(t) = \nabla f(t) := f(t) - f(t-1)$.

Definition 1.5. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* or *ld-continous* if it is continuous at left-dense points, and if its right-sided limit exists (finite) at right-dense points.

Definition 1.6. It can be shown that if f is ld-continuous then there is a function F, called a *∇-antiderivative*, such that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}$. We then define the *∇-integral* (∇-Cauchy integral) of f by

$$\int_{t_0}^{t} f(s)\ \nabla s = F(t) - F(t_0).$$

Theorem 1.7. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then we have the following:*

1. *If f is nabla-differentiable at t, then f is continuous at t.*
2. *If f is continuous at t and t is left-scattered, then f is nabla-differentiable at t with*
$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$
3. *If t is left-dense, then f is nabla-differentiable at t iff the limit*
$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$
exists as a finite number. In this case
$$f^{\nabla}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$
4. *If f is nabla-differentiable at t, then*
$$f^{\rho}(t) = f(t) - \nu(t) f^{\nabla}(t).$$

Other properties of both the ∇-derivative and the ∇-integral are analogous to the properties of the Δ-derivative and Δ-integral. For example, both differentiation and integration are linear operations, and there are product and quotient rules for differentiation, as well as integration by parts formulas. Readers interested in the specifics can find more details in [2].

2. PRELIMINARY RESULTS

We are interested in the second-order self-adjoint dynamic equation

$$Lx = 0 \quad \text{where} \quad Lx = [p(t)x^{\Delta}]^{\nabla} + q(t)x. \tag{2.1}$$

Here we assume that p is continuous, q is ld-continuous and that

$$p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

Define the set $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous and such that $[p(t)x^{\Delta}]^{\nabla} : \mathbb{T}^{\kappa}_{\kappa} \to \mathbb{R}$ is ld-continuous. A function $x \in \mathbb{D}$ is said to be a solution of $Lx = 0$ on $\mathbb{T}$ provided $Lx(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Since the equation we are interested in, equation (2.1), contains both Δ- and ∇-derivatives, we establish here some results regarding the relationship between these two types of derivatives on time scales.

One of the following results relies on L'Hôpital's rule. A version of L'Hôpital's rule involving Δ-derivatives is contained in [2]. We state its analog for ∇-derivatives here. As we may wish to use L'Hôpital's rule to evaluate a limit as $t \to \pm\infty$, we make the following definition.

Definition 2.1. Let $\varepsilon > 0$. If $\mathbb{T}$ is unbounded below, we define a *right neighborhood of* $-\infty$, denoted $R_\varepsilon(-\infty)$ by

$$R_\varepsilon(-\infty) = \left\{ t \in \mathbb{T} : t < -\frac{1}{\varepsilon} \right\}.$$

We next define a right neighborhood for points in $\mathbb{T}$.

Definition 2.2. Let $\varepsilon > 0$. For any right-dense $t_0 \in \mathbb{T}$, define a *right neighborhood of* t_0, denoted $R_\varepsilon(t_0)$, by

$$R_\varepsilon(t_0) := \{t \in \mathbb{T} : 0 < t - t_0 < \varepsilon\}.$$

Theorem 2.3 (L'Hôpital's Rule). *Assume f and g are ∇-differentiable on $\mathbb{T}$ and let $t_0 \in \mathbb{T} \cup \{-\infty\}$. If $t_0 \in \mathbb{T}$, assume t_0 is right-dense. Furthermore, assume*

$$\lim_{t \to t_0^+} f(t) = \lim_{t \to t_0^+} g(t) = 0,$$

and suppose there exists $\varepsilon > 0$ such that $g(t) g^\nabla(t) > 0$ for all $t \in R_\varepsilon(t_0)$. Then

$$\liminf_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)} \le \liminf_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)}.$$

Proof. Without loss of generality, assume $g(t)$ and $g^\nabla(t)$ are both strictly positive on $R_\varepsilon(t_0)$.

Let $\delta \in (0, \varepsilon]$, and let $a := \inf_{\tau \in R_\delta(t_0)} \frac{f^\nabla(\tau)}{g^\nabla(\tau)}$, $b := \sup_{\tau \in R_\delta(t_0)} \frac{f^\nabla(\tau)}{g^\nabla(\tau)}$. To complete the proof, it suffices to show

$$a \le \inf_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le \sup_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le b,$$

as we may then let $\delta \to 0$ to obtain the desired result.

We must be careful here, as either a or b could possibly be infinite. Note, however, that since $g^\nabla(\tau) > 0$ on $R_\delta(t_0)$, we have $a < \infty$. Similarly, $b > -\infty$. So our only concern is if $a = -\infty$ or $b = \infty$. But, if $a = -\infty$, we have immediately that

$$a \le \inf_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)},$$

as desired, and if $b = \infty$ we have immediately that

$$\sup_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le b,$$

as desired. Therefore, we may assume that both a and b are finite. Then

$$a g^\nabla(\tau) \le f^\nabla(\tau) \le b g^\nabla(\tau) \quad \text{for all} \quad \tau \in R_\delta(t_0),$$

and by a theorem of Guseinov and Kaymakçalan [3],

$$\int_t^s a g^\nabla(\tau)\, \nabla\tau \le \int_t^s f^\nabla(\tau)\, \nabla\tau \le \int_t^s b g^\nabla(\tau)\, \nabla\tau \quad \text{for all} \quad s, t \in R_\delta(t_0),\ t < s.$$

Integrating, we see that

$$ag(s) - ag(t) \le f(s) - f(t) \le bg(s) - bg(t) \quad \text{for all} \quad s, t \in R_\delta(t_0),\ t < s.$$

Letting $t \to t_0^+$, we get

$$ag(s) \le f(s) \le bg(s) \quad \text{for all} \quad s \in R_\delta(t_0),$$

and thus

$$a \le \inf_{s \in R_\delta(t_0)} \frac{f(s)}{g(s)} \le \sup_{s \in R_\delta(t_0)} \frac{f(s)}{g(s)} \le b.$$

Then, by the discussion above, the proof is complete. □

Remark 2.4. Although the preceding theorem is only stated in terms of one-sided limits, an analogous result can be established if the limit is taken from the other direction. Left neighborhoods of ∞ or of points in $\mathbb{T}$ are defined in similar manner to right neighborhoods. To apply L'Hôpital's rule using a left-sided limit, t_0 must be left-dense (or ∞ if $\mathbb{T}$ is unbounded above), and gg^∇ must be strictly negative on some left neighborhood of t_0.

In order to determine when the two types of derivatives may be interchanged, we need to consider some of the points in our time scale separately, so let

$$A := \{t \in \mathbb{T} \mid t \text{ is left-dense and right-scattered}\}, \quad \mathbb{T}_A := \mathbb{T} \setminus A.$$

Additionally, let

$$B := \{t \in \mathbb{T} \mid t \text{ is right-dense and left-scattered}\}, \quad \mathbb{T}_B := \mathbb{T} \setminus B.$$

The following lemma is very easy to prove, and we omit the proof here.

Lemma 2.5. *If $t \in \mathbb{T}_A$ then $\sigma(\rho(t)) = t$. If $t \in \mathbb{T}_B$ then $\rho(\sigma(t)) = t$.*

Theorem 2.6. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and f^Δ is rd-continuous on $\mathbb{T}^\kappa$ then f is ∇-differentiable on $\mathbb{T}_\kappa$, and*

$$f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) & t \in \mathbb{T}_A \\ \lim_{s \to t^-} f^\Delta(s) & t \in A. \end{cases}$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa$ and g^∇ is ld-continuous on $\mathbb{T}_\kappa$ then g is Δ-differentiable on $\mathbb{T}^\kappa$, and

$$g^\Delta(t) = \begin{cases} g^\nabla(\sigma(t)) & t \in \mathbb{T}_B \\ \lim_{s \to t^+} g^\nabla(s) & t \in B. \end{cases}$$

Proof. We will only prove the first statement. The proof of the second statement is similar. First, assume $t \in \mathbb{T}_A$. Then there are two cases: Either

1. t is left-scattered, or
2. t is both left-dense and right-dense.

Case 1: Suppose t is left-scattered and f is Δ-differentiable on $\mathbb{T}^\kappa$. Then f is continuous at t, and is therefore ∇-differentiable at t. Next, note that $\rho(t)$ is right-scattered, and

$$\begin{aligned} f^\Delta(\rho(t)) &= \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)} \\ &= \frac{f(t) - f(\rho(t))}{t - \rho(t)} \\ &= f^\nabla(t). \end{aligned}$$

Case 2: Now, suppose t is both left-dense and right-dense, and $f : \mathbb{T} \to \mathbb{R}$ is continuous on $\mathbb{T}$ and Δ-differentiable at t. Since t is right-dense and f is Δ-differentiable at t, we have that

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists. But t is left-dense as well, so this expression also defines $f^\nabla(t)$, and we see that

$$\begin{aligned} f^\nabla(t) &= \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \\ &= f^\Delta(t) \\ &= f^\Delta(\rho(t)). \end{aligned}$$

So, we have established the desired result in the case where $t \in \mathbb{T}_A$.

Now suppose $t \in A$. Then t is left-dense. Hence $f^\nabla(t)$ exists provided

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists.

As t is right-scattered, we need only consider the limit as $s \to t$ from the left. Then we apply L'Hôpital's rule [2], differentiating with respect to s to get

$$\lim_{s \to t^-} \frac{f(t) - f(s)}{t - s} = \lim_{s \to t^-} \frac{-f^\Delta(s)}{-1} = \lim_{s \to t^-} f^\Delta(s).$$

Since we have assumed that f^Δ is rd-continuous, this limit exists. Hence f is ∇-differentiable, and $f^\nabla(t) = \lim_{s \to t^-} f^\Delta(t)$, as desired. $\square$

Corollary 2.7. *If $t_0 \in \mathbb{T}$, and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$ then $\int_{t_0}^t f(\tau)\Delta\tau$ is ∇-differentiable on $\mathbb{T}$ and*

$$\left[\int_{t_0}^t f(\tau)\Delta\tau\right]^\nabla = \begin{cases} f(\rho(t)) & \text{if } t \in \mathbb{T}_A \\ \lim_{s \to t^-} f(s) & t \in A. \end{cases}$$

If $t_0 \in \mathbb{T}$, and $g : \mathbb{T} \to \mathbb{R}$ is ld-continuous on $\mathbb{T}$ then $\int_{t_0}^t g(\tau)\nabla\tau$ is Δ-differentiable on $\mathbb{T}$ and

$$\left[\int_{t_0}^t g(\tau)\nabla\tau\right]^\Delta = \begin{cases} g(\sigma(t)) & \text{if } t \in \mathbb{T}_B \\ \lim_{s \to t^+} g(s) & t \in B. \end{cases}$$

The following corollary was previously established by Atıcı and Guseinov in their work [1].

Corollary 2.8. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and if f^Δ is continuous on $\mathbb{T}^\kappa$, then f is ∇-differentiable on $\mathbb{T}_\kappa$ and*

$$f^\nabla(t) = f^{\Delta\rho}(t) \quad \textit{for } t \in \mathbb{T}_\kappa.$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}^\kappa$ and if g^∇ is continuous on $\mathbb{T}_\kappa$, then g is Δ-differentiable on $\mathbb{T}^\kappa$ and

$$g^\Delta(t) = g^{\nabla\sigma}(t) \quad \textit{for } t \in \mathbb{T}^\kappa.$$

3. ABEL'S FORMULA AND REDUCTION OF ORDER

We begin this section by looking at the Lagrange Identity for the dynamic equation (2.1). We establish several corollaries and related results, including Abel's formula and its converse. We conclude the section with a reduction of order theorem. Some of the results in this section are due to Atıcı and Guseinov. Specifically, Theorem 3.1 and Corollary 3.5 were previously established in their work [1]. Our conditions on p and q are less restrictive than Atıcı and Guseinov's, and our domain of interest, $\mathbb{D}$, is defined more broadly. In spite of this, however, many of the proofs contained in [1] remain valid. As this is the case, we have omitted the proofs of some of the following theorems, and refer the reader to Atıcı and Guseinov's work.

Theorem 3.1. *If $t_0 \in \mathbb{T}$, and x_0 and x_1 are given constants, then the initial value problem*

$$Lx = 0, \quad x(t_0) = x_0, \quad x^\Delta(t_0) = x_1$$

has a unique solution, and this solution exists on all of $\mathbb{T}$.

Definition 3.2. If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the *Wronskian* of x and y, denoted $W(x, y)(t)$ is defined by

$$W(x,y)(t) = \begin{vmatrix} x(t) & y(t) \\ x^\Delta(t) & y^\Delta(t) \end{vmatrix} \quad \text{for } t \in \mathbb{T}^\kappa.$$

Definition 3.3. If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the *Lagrange bracket* of x and y is defined by

$$\{x; y\}(t) = p(t)W(x,y)(t) \quad \text{for } t \in \mathbb{T}^\kappa.$$

Theorem 3.4 (Lagrange Identity). *If $x, y \in \mathbb{D}$, then*

$$x(t)Ly(t) - y(t)Lx(t) = \{x; y\}^\nabla(t) \quad \textit{for } t \in \mathbb{T}^\kappa_\kappa.$$

Proof. Let $x, y \in \mathbb{D}$. We have

$$\begin{aligned}
\{x; y\}^{\nabla} &= [pW(x,y)]^{\nabla} \\
&= [xpy^{\Delta} - ypx^{\Delta}]^{\nabla} \\
&= x^{\nabla}p^{\rho}y^{\Delta\rho} + x[py^{\Delta}]^{\nabla} - y^{\nabla}p^{\rho}x^{\Delta\rho} - y[px^{\Delta}]^{\nabla} \\
&= x^{\nabla}p^{\rho}y^{\nabla} + x[py^{\Delta}]^{\nabla} - y^{\nabla}p^{\rho}x^{\nabla} - y[px^{\Delta}]^{\nabla} \\
&= x[py^{\Delta}]^{\nabla} - y[px^{\Delta}]^{\nabla} \\
&= x([py^{\Delta}]^{\nabla} + qy) - y([px^{\Delta}]^{\nabla} + qx) \\
&= xLy - yLx,
\end{aligned}$$

where we have made use of the fact that x^{Δ} and y^{Δ} are continuous and applied Corollary 2.8. □

Corollary 3.5 (Abel's Formula)**.** *If x, y are solutions of* (2.1) *then*

$$W(x,y)(t) = \frac{C}{p(t)} \quad \textit{for } t \in \mathbb{T}^{\kappa},$$

where C is a constant.

Definition 3.6. Define the *inner product* of x and y on $[a,b]$ by

$$\langle x, y\rangle := \int_a^b x(t)y(t)\nabla t.$$

Corollary 3.7 (Green's Formula)**.** *If $x, y \in \mathbb{D}$ then*

$$\langle x, Ly\rangle - \langle Lx, y\rangle = [p(t)W(x,y)]_a^b.$$

Theorem 3.8 (Converse of Abel's Formula)**.** *Assume u is a solution of* (2.1) *with $u(t) \neq 0$ for $t \in \mathbb{T}$. If $v \in \mathbb{D}$ satisfies*

$$W(u,v)(t) = \frac{C}{p(t)},$$

then v is also a solution of (2.1)*.*

Proof. Suppose that u is a solution of (2.1) with $u(t) \neq 0$ for any t, and assume that $v \in \mathbb{D}$ satisfies $W(u,v)(t) = \frac{C}{p(t)}$. Then by Theorem 3.4, we have

$$\begin{aligned}
u(t)Lv(t) - v(t)Lu(t) &= \{u; v\}^{\nabla}(t) \\
u(t)Lv(t) &= [p(t)W(u,v)(t)]^{\nabla} \\
&= [p(t)\frac{C}{p(t)}]^{\nabla} \\
&= C^{\nabla} \\
&= 0.
\end{aligned}$$

As $u(t) \neq 0$ for any t, we can divide through by it to get

$$Lv(t) = 0 \quad \text{for } t \in \mathbb{T}^{\kappa}_{\kappa}.$$

Hence v is a solution of (2.1) on $\mathbb{T}$. $\square$

Theorem 3.9 (Reduction of Order)**.** *Let $t_0 \in \mathbb{T}^{\kappa}$, and assume u is a solution of* (2.1) *with $u(t) \neq 0$ for any t. Then a second, linearly independent solution, v, of* (2.1) *is given by*

$$v(t) = u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s$$

for $t \in \mathbb{T}$.

Proof. By Theorem 3.8, we need only show that $v \in \mathbb{D}$ and that $W(u, v)(t) = \frac{C}{p(t)}$ for some constant C. Consider first

$$\begin{aligned}
W(u,v)(t) &= u(t)v^{\Delta}(t) - v(t)u^{\Delta}(t) \\
&= u(t)\left[u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s + \frac{u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)}\right] \\
&\quad - u^{\Delta}(t)u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s \\
&= u(t)u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s + \frac{u(t)u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)} \\
&\quad - u(t)u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s \\
&= \frac{1}{p(t)}.
\end{aligned}$$

Here we have $C = 1$. It remains to show that $v \in \mathbb{D}$. We have that

$$\begin{aligned}
v^{\Delta}(t) &= u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s + \frac{u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)} \\
&= u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s + \frac{1}{p(t)u(t)}.
\end{aligned}$$

Since $u \in \mathbb{D}, u(t) \neq 0$ and p is continuous, we have that v^{Δ} is continuous. Next, consider

$$\begin{aligned}
[p(t)v^{\Delta}(t)]^{\nabla} &= \left[p(t)u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s\right]^{\nabla} + \left[\frac{1}{u(t)}\right]^{\nabla} \\
&= [p(t)u^{\Delta}(t)]^{\nabla} \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s \\
&\quad + p^{\rho}(t)u^{\Delta\rho}(t) \left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s\right]^{\nabla} - \frac{u^{\Delta}(t)}{u(t)u^{\rho}(t)}.
\end{aligned}$$

Now, the first and last terms are ld-continuous. It is not as clear that the center term is ld-continuous. Specifically, we are concerned about whether or not the expression

$$\left[\int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)}\Delta s\right]^{\nabla}$$

is ld-continuous. Note that the integrand is rd-continuous. Hence Corollary 2.7 applies and yields

$$\left[\int_{t_0}^{t} \frac{1}{p(\tau)u(\tau)u^{\sigma}(\tau)}\Delta \tau\right]^{\nabla} = \begin{cases} \frac{1}{p^{\rho}(t)u^{\rho}(t)u^{\sigma\rho}(t)} & \text{if } t \in \mathbb{T}_A \\ \lim_{s\to t^-} \frac{1}{p(s)u(s)u^{\sigma}(s)} & t \in A. \end{cases}$$

Simplification of this expression gives

$$\left[\int_{t_0}^{t} \frac{1}{p(\tau)u(\tau)u^{\sigma}(\tau)}\Delta \tau\right]^{\nabla} = \frac{1}{p^{\rho}(t)u^{\rho}(t)u(t)} \quad \text{for } t \in \mathbb{T}.$$

This function is ld-continuous, and so we have that $v \in \mathbb{D}$. Hence by Theorem 3.8, v is also a solution of (2.1). Finally, note that as $W(u,v)(t) = \frac{1}{p(t)} \neq 0$ for any t, u and v are linearly independent. □

4. OSCILLATION AND DISCONJUGACY

In this section, we establish results concerning generalized zeros of solutions of (2.1), and examine disconjugacy and oscillation of solutions.

Definition 4.1. We say that a solution, x, of (2.1) has a *generalized zero* at t if

$$x(t) = 0$$

or, if t is left-scattered and

$$x(\rho(t))x(t) < 0.$$

Definition 4.2. We say that (2.1) is *disconjugate* on an interval $[a,b]$ if the following hold.

1. If x is a nontrivial solution of (2.1) with $x(a) = 0$, then x has no generalized zeros in $(a,b]$.
2. If x is a nontrivial solution of (2.1) with $x(a) \neq 0$, then x has at most one generalized zero in $(a,b]$.

We will investigate oscillation of (2.1) as t approaches the supremum of the time scale. Let $\omega = \sup \mathbb{T}$. If $\omega < \infty$, we assume $\rho(\omega) = \omega$. Furthermore, if $\omega < \infty$, we allow the possibility that ω is a singular point for p or q.

Definition 4.3. Let $\omega = \sup \mathbb{T}$ be as described above, and let $a \in \mathbb{T}$. We say that (2.1) is *oscillatory* on $[a,\omega)$ if every nontrivial real-valued solution has infinitely many generalized zeros in $[a,\omega)$. We say (2.1) is *nonoscillatory* if it is not oscillatory.

The following Lemma is a direct consequence of the definition of nonoscillatory.

Lemma 4.4. *Let $\omega = \sup \mathbb{T}$ be as described above, and let $a \in \mathbb{T}$. Then if* (2.1) *is nonoscillatory on $[a, \omega)$, there is some $t_0 \in \mathbb{T}$, $t_0 \geq a$, such that* (2.1) *has a positive solution on $[t_0, \omega)$.*

Theorem 4.5 (Sturm Separation Theorem)**.** *Let u and v be linearly independent solution of* (2.1)*. Then u and v have no common zeros in $\mathbb{T}^\kappa$. If u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $(t_1, t_2]$. If u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $[t_1, t_2]$.*

Proof. If u and v have a common zero at $t_0 \in \mathbb{T}^\kappa$, then

$$W(u,v)(t_0) = \begin{vmatrix} u(t_0) & v(t_0) \\ u^\Delta(t_0) & v^\Delta(t_0) \end{vmatrix} = 0.$$

Hence u and v are linearly dependent.

Now suppose u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$. Without loss of generality, we may assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1, $u(t) > 0$ on (t_1, t_2), and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (2.1) with no generalized zero in $(t_1, t_2]$. Without loss of generality, $v(t) > 0$ on $[t_1, t_2]$.

Then on $[t_1, t_2]$,

$$\left(\frac{u}{v}\right)^\Delta (t) = \frac{v(t)u^\Delta(t) - u(t)v^\Delta(t)}{v(t)v^\sigma(t)} = \frac{C}{p(t)v(t)v^\sigma(t)},$$

which is of one sign on $[t_1, t_2)$. Thus $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Fix $t_3 \in (t_1, t_2)$. Note that

$$\frac{u(t_1)}{v(t_1)} = 0, \text{ and } \frac{u(t_3)}{v(t_3)} > 0.$$

But

$$\frac{u(t_2)}{v(t_2)} \leq 0,$$

which contradicts the fact that $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Hence v must have a generalized zero in $(t_1, t_2]$.

Finally, suppose u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$. Assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1. If $u(t_1) = 0$, we are in the previous case, so assume $u(t_1) \neq 0$. Then, as u has a generalized zero at t_1, we have that t_1 is left-scattered. Without loss of generality, we may assume $u(t) > 0$ on $[t_1, t_2)$, $u(\rho(t_1)) < 0$ and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (2.1) with no generalized zero in $[t_1, t_2)$. Without loss of generality, $v(t) > 0$ on

$[t_1, t_2]$, and $v(\rho(t_1)) > 0$. In a similar fashion to the previous case, we apply Abel's formula to get that $\frac{u}{v}$ is monotone on $[\rho(t_1), t_2]$. But

$$\frac{u(\rho(t_1))}{v(\rho(t_1))} < 0, \quad \frac{u(t_1)}{v(t_1)} > 0, \quad \text{and} \quad \frac{u(t_2)}{v(t_2)} \leq 0,$$

which is a contradiction. Hence v must have a generalized zero in $[t_1, t_2]$. □

Theorem 4.6. *If* (2.1) *has a positive solution on an interval* $\mathcal{I} \subset \mathbb{T}$ *then* (2.1) *is disconjugate on* $\mathcal{I}$. *Conversely, if* $a, b \in \mathbb{T}^\kappa_\kappa$ *and* (2.1) *is disconjugate on* $[\rho(a), \sigma(b)] \subset \mathbb{T}$, *then* (2.1) *has a positive solution on* $[\rho(a), \sigma(b)]$.

Proof. If (2.1) has a positive solution, u on $\mathcal{I} \subset \mathbb{T}$, then disconjugacy follows from the Sturm separation theorem.

Conversely, if (2.1) is disconjugate on the compact interval $[\rho(a), \sigma(b)]$, then let u, v be the solutions of (2.1) satisfying $u(\rho(a)) = 0, u^\Delta(\rho(a)) = 1$ and $v(\sigma(b)) = 0, v^\Delta(b) = -1$. Since (2.1) is disconjugate on $[\rho(a), \sigma(b)]$, we have that $u(t) > 0$ on $(\rho(a), \sigma(b)]$, and $v(t) > 0$ on $[\rho(a), \sigma(b))$. Then

$$x(t) = u(t) + v(t)$$

is the desired positive solution. □

Theorem 4.7 (Pólya Factorization)**.** *If* (2.1) *has a positive solution, u, on an interval* $\mathcal{I} \subset \mathbb{T}$, *then for any* $x \in \mathbb{D}$, *we get the* Pólya factorization

$$Lx = \alpha_1(t)\{\alpha_2[\alpha_1 x]^\Delta\}^\nabla(t) \text{ for } t \in \mathcal{I},$$

where

$$\alpha_1 := \frac{1}{u} > 0 \quad \text{on } \mathcal{I},$$

and

$$\alpha_2 := puu^\sigma > 0 \quad \text{on } \mathcal{I}.$$

Proof. Assume that u is a positive solution of (2.1) on $\mathcal{I}$, and let $x \in \mathbb{D}$. Then by the Lagrange Identity (Theorem 3.4),

$$\begin{aligned} u(t)Lx(t) - x(t)Lu(t) &= \{u; x\}^\nabla(t) \\ Lx(t) &= \frac{1}{u(t)}\{pW(u,x)\}^\nabla(t) \\ &= \frac{1}{u(t)}\left\{puu^\sigma\left[\frac{x}{u}\right]^\Delta\right\}^\nabla(t) \\ &= \alpha_1(t)\{\alpha_2\,[\alpha_1 x]^\Delta\}^\nabla(t), \end{aligned}$$

for $t \in \mathcal{I}$, where α_1 and α_2 are as described in the theorem. □

Theorem 4.8 (Trench Factorization)**.** *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, assume $\rho(\omega) = \omega$. If (2.1) is nonoscillatory on $[a, \omega)$, then there is $t_0 \in \mathbb{T}$ such that for any $x \in \mathbb{D}$, we get the* Trench factorization

$$Lx(t) = \beta_1(t)\{\beta_2[\beta_1 x]^{\Delta}\}^{\nabla}(t)$$

for $t \in [t_0, \omega)$, where $\beta_1, \beta_2 > 0$ on $[t_0, \omega)$, and

$$\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t = \infty.$$

Proof. Since (2.1) is nonoscillatory on $[a, \omega)$, (2.1) has a positive solution, u on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$. Then by Theorem 4.7, Lx has a Pólya factorization on $[t_0, \omega)$. Thus there are functions α_1 and α_2 such that

$$Lx(t) = \alpha_1(t)\{\alpha_2[\alpha_1 x]^{\Delta}\}^{\nabla}(t) \text{ for } t \in [t_0, \omega),$$

defined as described in the preceding theorem. Now, if

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)} \Delta t = \infty,$$

then take $\beta_1(t) = \alpha_1(t)$, and $\beta_2(t) = \alpha_2(t)$, and we are done. Therefore, assume that

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)} \Delta t < \infty.$$

In this case, let

$$\beta_1(t) = \frac{\alpha_1(t)}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \text{ and } \beta_2(t) = \alpha_2(t) \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s$$

for $t \in [t_0, \omega)$. Note that as $\alpha_1, \alpha_2 > 0$, we have $\beta_1, \beta_2 > 0$ as well. Also,

$$\begin{aligned}
\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t &= \lim_{b \to \omega} \int_{t_0}^{b} \frac{1}{\alpha_2(t) \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \Delta t \\
&= \lim_{b \to \omega} \int_{t_0}^{b} \left[\frac{1}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \right]^{\Delta} \Delta t \\
&= \lim_{b \to \omega} \left[\frac{1}{\int_b^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \right] \\
&= \infty.
\end{aligned}$$

Now let $x \in \mathbb{D}$. Then

$$[\beta_1 x]^{\Delta}(t) = \left[\frac{\alpha_1(t) x(t)}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \right]^{\Delta} = \frac{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s [\alpha_1(t) x(t)]^{\Delta} + \alpha_1(t) x(t) \frac{1}{\alpha_2(t)}}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s}$$

for $t \in [t_0, \omega)$. So we get

$$\beta_2(t)[\beta_1(t) x]^{\Delta} = \alpha_2(t)[\alpha_1(t) x(t)]^{\Delta} \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s + \alpha_1(t) x(t)$$

for $t \in [t_0, \omega)$. Taking the ∇ -derivative of both sides gives

$$\begin{aligned}\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla &= \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s \\ &\quad + \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\rho \left[\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s\right]^\nabla \\ &\quad + [\alpha_1(t)x(t)]^\nabla\end{aligned}$$

for $t \in [t_0, \omega)$. We now claim that the last two terms in this expression cancel. To see this, put the expression back in terms of our positive solution u, and consider $t \in \mathbb{A}$ and $t \in \mathbb{T}_A$ separately. Careful application of Theorem 2.6 then shows that these terms do, in fact cancel, and we get

$$\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s.$$

It then follows that

$$\beta_1(t)\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \alpha_1(t)\left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla = Lx(t),$$

for $t \in [t_0, \omega)$ and the proof is complete. □

Theorem 4.9 (Recessive and Dominant Solutions). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$ the we assume $\rho(\omega) = \omega$. If (2.1) is nonoscillatory on $[a, \omega)$, then there is a solution, u, called a recessive solution at ω, such that u is positive on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$, and if v is any second, linearly independent solution, called a dominant solution at ω, the following hold.*

1. $\lim_{t\to\omega^-} \frac{u(t)}{v(t)} = 0$
2. $\int_{t_0}^\omega \frac{1}{p(t)u(t)u^\sigma(t)}\Delta t = \infty$
3. $\int_b^\omega \frac{1}{p(t)v(t)v^\sigma(t)}\Delta t < \infty$ *for $b < \omega$, sufficiently close, and*
4. $\frac{p(t)v^\Delta(t)}{v(t)} > \frac{p(t)u^\Delta(t)}{u(t)}$ *for $t < \omega$, sufficiently close.*

The recessive solution, u, is unique, up to multiplication by a nonzero constant.

Proof. The proof of this theorem is directly analogous to the standard proof used in the differential equations case. See, for example, [5]. □

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