51. Let $S = \{[a \quad b \quad c \quad d]^T \in \mathbb{R}^4 : a+b+c+d=0\}$. Find $S^\perp$.

52. Give an example of a 3×2-matrix A with
   (a) $\mathcal{N}(A) \neq \{0\}$ and check that $\mathcal{N}(A) = \mathcal{N}(A^TA)$;
   (b) $\mathcal{N}(A) = \{0\}$ and check that A^TA is invertible.

53. Let A be an $m \times n$-matrix. Prove

   (a) $\mathcal{N}(A) = \mathcal{N}(A^TA)$ (b) If $\mathcal{N}(A) = \{0\}$, then A^TA is invertible.

54. Use the Cauchy-Schwarz inequality to compare the algebraic mean $\frac{1}{2}(x+y)$ of two positive numbers x and y with the geometric mean $\sqrt{xy}$.

55. Consider a company that prints books.
   (a) If no books are printed, the costs for the company are \$2000. If 100 books are printed, the costs are \$5000. For 200 and 300 books the costs are \$8000 and \$11000, respectively. Draw the data into a coordinate system, using the number of books divided by 100 on the x-axis and the costs in Dollars divided by 1000 on the y-axis. Can you find a line such that each of the points is on that line? (Will work unless you did a mistake.) Give the equation $l(x)$ of this line. Use it to estimate the costs for producing 400 books.
   (b) Same problem as in (a), but the data are now as follows: For 0, 100, 200, and 300 books the costs are \$1500, \$6200, \$8700, and \$11000, respectively. Plot the data in a coordinate system as in (a). Try to find a line through them (won't work unless you did a mistake). Use the line $l(x)$ from (a) and compute the sum $(1.5-l(0))^2+(6.2-l(1))^2+(8.7-l(2))^2+(11-l(3))^2$. What is this sum geometrically? Now consider the line $\tilde{l}(x) = 3x+2.1$. Evaluate the corresponding sum for the line $\tilde{l}(x)$. Now find the line that has the smallest possible such sum (give this sum). Use this line to predict the cost of producing 400 books.

56. On 10 stock exchange days the cash courses x and y of the shares of two automobile companies read as follows: 420, 429, 445, 418, 431, 459, 451, 465, 449, 473 (for x) and 495, 506, 516, 475, 493, 531, 537, 554, 547, 565 (for y). Plot the data into a coordinate system. Are they on a line? (Most likely not.) Find the line that fits the data best using the method from the previous problem. Use this line to predict the course of y when $x = 455$.

57. For the following matrices, split a given vector x uniquely into a sum of two vectors $x_1 + x_2$, where $x_1 \in \mathcal{R}(A)$ and $x_2 \in \mathcal{N}(A^T)$:

$$\text{(a) } A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{(b) } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{(c) } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{(d) } A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

58. Calculate $A^\dagger = \lim_{\varepsilon \to 0^+}(A^TA+\varepsilon I)^{-1}A^T$ for each of the matrices A from the previous problem. Verify that the orthogonal projection of x onto $\mathcal{R}(A)$ (i.e., the x_1 from the previous problem) is equal to $AA^\dagger x$.

59. Work on problems 3.4.1, 3.4.6, 3.4.13, 3.4.14 from the textbook.