1. Solve the system $2u + v = 8$, $4u - \frac{3}{2}v = 9$ using each of the four methods presented in class.

2. Find a polynomial of degree two whose graph goes through the points:

   (a) $(1, -1)$, $(2, 3)$, and $(3, 13)$;

   (b) $(1, s_1)$, $(2, s_2)$, and $(3, s_3)$, where $s_1, s_2, s_3 \in \mathbb{R}$.

3. For the following systems of equations, do the following: Rewrite the systems as an equation $Ax = b$, do Gaussian Elimination and write down the elementary matrices needed, find the LDU Decomposition of A, find c such that $Lc = b$ and finally find x such that $DUx = c$:

   (a) $2u + 4v = 3$, $3u + 7v = 2$;

   (b) $3u + 5v + 3w = 25$, $7u + 9v + 19w = 65$, $-4u + 5v + 11w = 5$;

   (c) $u + 2v + 3w = 39$, $u + 3v + 2w = 34$, $3u + 2v + w = 26$;

   (d) $u + 3v + 5w = 1$, $3u + 12v + 18w = 1$, $5u + 18v + 30w = 1$;

   (e) $\alpha u + \beta v = 1$, $\beta u + \gamma v = 1$ (where $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha(\alpha\gamma - \beta^2) \neq 0$).

4. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

   (a) Compute AC, CB, ACB, A^2, B^2, CC^T.

   (b) Find A^n and B^n for all $n \in \mathbb{N}$ (prove your claim using the Principle of Mathematical Induction).

   (c) Show that A is not invertible. Also show that B is invertible and find B^{-1}.

5. Prove Proposition 1.1(b), i.e., matrix operations are distributive.

6. Use the Gauss-Jordan method to find the inverses of:

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

   (b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$;

   (c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

7. Find examples of 2×2-matrices with:

   (a) $A^2 = -I$ (A having only real entries);

   (b) $B^2 = 0$ (but $B \neq 0$);

   (c) $CD = -DC$ (but $CD \neq 0$);

   (d) $EF = 0$ (neither E nor F having any zero entries);

   (e) $AB = AC$ but $B \neq C$;

(f) $A+B$ is not invertible but A and B are;

(g) $A+B$ is invertible but A and B are not;

(h) A and B are symmetric but AB is not.

8. Prove Proposition 1.2, i.e., the LDU Factorization is unique.

9. Let A be any matrix. Show that AA^T and A^TA are both symmetric.

10. If A, B, and $A+B$ are invertible, show that $A^{-1}+B^{-1}$ is invertible and find a formula for its inverse in terms of A, B, $A+B$ and their inverses.

11. Consider an $m \times m$-matrix A whose entries are all nonnegative. Suppose the ith column sum of A is r_i and let $r = \max\{r_i |\ 1 \leq i \leq m\}$. Assume $r < 1$.

    (a) Show that all entries of A^n are less than r^n, for all $n \in \mathbb{N}$.

    (b) Show that $\lim_{n\to\infty} A^n = 0$ (meaning that all entries of A^n tend to 0 as $n \to \infty$).

    (c) Show that $\lim_{n\to\infty} \sum_{k=0}^{n} A^k$ exists (again considering entries).

    (d) Compute and simplify $(I-A)\sum_{k=0}^{n} A^k$ for all $n \in \mathbb{N}$.

    (e) Calculate $\lim_{n\to\infty} \sum_{k=0}^{n} A^k$.