48. Given is the second order difference equation $ay_{n+2} + by_{n+1} + cy_n = 0$.
    (a) If $y_n = 2^{n-1} - 5^{n+1}$ for all natural numbers n, find a, b, and c.
    (b) If $y_0 = 1$, $y_1 = 2$, $y_2 = 5$, $y_3 = 12$, $y_4 = 29$, $y_5 = 70$, $y_6 = 169$, find a, b, and c. Also determine y_n for each natural number n.
49. The *Casoratian* of two sequences f and g is defined by $C_k = \det \begin{pmatrix} f_k & g_k \\ \Delta f_k & \Delta g_k \end{pmatrix}$.
    (a) Find the Casoratian of 2^k and 3^k.
    (b) Find the Casoratian of 4^k and $k4^k$.
    (c) Find two solutions of $\Delta^2 y_k = 0$ such that their Casoratian is one.
    (d) Show that the Casoratian of any two solutions of $\Delta^2 y_k + q_k y_{k+1} = 0$ is constant.
    (e) Find the Casoratian of any two solutions of $\Delta(p_k \Delta y_k) + q_k y_{k+1} = 0$ if $p_k \neq 0$.
    (f) Write the difference equation of the Fibonacci sequence in the form given in (e).
50. For the following equations, find one solution y_1 using the characteristic polynomial, and then try to find a second solution by trying $y_2(t) = v(t)y_1(t)$ for some function v that needs to be determined. Make sure that the Wronskian of y_1 and y_2 is not zero. Then find the solution y with $y(0) = 0$ and $y'(0) = 1$.
    (a) $y'' - 2y' + y = 0$;
    (b) $y'' - 4y' + 4y = 0$;
    (c) $y'' - 6y' + 9y = 0$.
51. Based on the previous problem, suggest a general method how to find a fundamental system of a second order linear differential equation if the characteristic polynomial has exactly one real zero.
52. Based on the previous problem, suggest a general method how to find a fundamental system (i.e., two solutions with nonzero Casoratian) of a second order linear difference equation if the characteristic polynomial has exactly one real zero. You first might want to calculate some examples.
53. For each of the following initial value problems, find the solution.
    (a) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$;
    (b) $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$;
    (c) $y'' - 2y' + 5y = 0$, $y(\frac{\pi}{2}) = 0, y'(\frac{\pi}{2}) = 2$.
54. Work on Problems 23–27 of Section 3.4 in the textbook.
55. For the following equations, find one particular solution (hint: Try ae^{bt} or $a\sin(bt) + c\cos(dt)$).
    (a) $y'' - 2y' - 3y = 3e^{2t}$;
    (b) $y'' + 2y' + 4y = 2e^{-t}$;
    (c) $y'' + 2y' + 5y = 3\sin(2t)$;
    (d) $y'' + y = 3\sin(3t) + 4\cos(3t)$.