12. We define the harmonic numbers as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

(a) Find H_1, H_2, H_3, H_4, H_5 as fractions $\frac{m}{n}$ with $m, n \in \mathbb{Z}$.

(b) What is $H_{n+1} - H_n$ for $n \in \mathbb{N}$?

(c) Prove by mathematical induction that $H_{2^n} \le 1 + n$ holds for all $n \in \mathbb{N}_0$.

(d) Prove by mathematical induction that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$ holds for all $n \in \mathbb{N}$.

13. By experimenting with some values of n, guess a formula for the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)},$$

and then use mathematical induction to verify your formula.

14. Show that $\prod_{k=0}^{n} \left(1 + x^{(2^k)}\right) = \frac{1 - x^{(2^{n+1})}}{1-x}$ is true for all $x \neq 1$ and all $n \in \mathbb{N}$.

15. Work on problems 1–16 of Section 2.1 in the textbook.

16. Let $a_n = \frac{1}{n} - \frac{1}{n+1}$, $n \in \mathbb{N}$.

(a) Find a_1, a_2, a_3, a_4, and a_5.

(b) Find $\sum_{k=1}^{5} a_k$, $\prod_{k=1}^{5} a_k$, $\sum_{k=1}^{100} a_k$, and $\prod_{k=1}^{100} a_k$.

(c) Let $m, n \in \mathbb{N}$. Find $\sum_{k=m}^{n} a_k$ and $\prod_{k=m}^{n} a_k$.

(d) Is a increasing or decreasing?

(e) Is a bounded above or bounded below?

17. Let $a_1 = 3$ and $a_n = 3 + a_{n-1}$ for $n \in \mathbb{N} \setminus \{1\}$.

(a) Find a_1, a_2, a_3, a_4, and a_5.

(b) Find $\sum_{k=1}^{5} a_k$ and $\prod_{k=1}^{5} a_k$.

(c) Let $n \in \mathbb{N}$. Find $\sum_{k=1}^{n} a_k$ and $\prod_{k=1}^{n} a_k$.

(d) Is a increasing or decreasing or bounded above or bounded below?