58. Show that the Dirichlet, Neumann, Robin, and periodic boundary conditions are symmetric. For which conditions on $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are the conditions $f(b) = \alpha f(a) + \beta f'(a)$, $f'(b) = \gamma f(a) + \delta f'(a)$ symmetric?

59. Consider an eigenvalue problem $f'' + \lambda f = 0$ with symmetric boundary conditions.
    (a) Show that if $f(b)f'(b) - f(a)f'(a) \le 0$ for all $f : [a, b] \to \mathbb{R}$ satisfying the boundary conditions, then there is no negative eigenvalue.
    (b) Show that the condition in (a) is satisfied for Dirichlet, Neumann, and periodic boundary conditions. In which cases is it satisfied for Robin conditions?

60. Let V be a complex vector space. An inner product on V is a mapping $(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that for all $x, y, z \in V$ and all $\lambda \in \mathbb{C}$: $(x, y) = \overline{(y, x)}$, $(\lambda x, y) = \lambda(x, y)$, $(x + y, z) = (x, z) + (y, z)$, and $(x, x) > 0$ if $x \neq 0$.
    (a) Prove $(x, y + z) = (x, y) + (x, z)$ and $(x, \lambda y) = \overline{\lambda}(x, y)$.
    (b) Prove $\|x\| \ge 0$ and $\|x\| = 0$ iff $x = 0$ and $\|\lambda x\| = |\lambda| \, \|x\|$.
    (c) Show that $(x, y) = x^T \overline{y}$ is an inner product on $\mathbb{C}^n$.
    (d) Show that $(f, g) = \int_a^b f(x)\overline{g(x)}\mathrm{d}x$ is an inner product on the vector space of all continuous complex-valued functions on $[a, b]$.

61. Let V be a real vector space with inner product $(\cdot, \cdot)$. We call $x, y \in V$ orthogonal (write $x \perp y$) if $(x, y) = 0$. Also, we put $\|x\| = \sqrt{(x, x)}$. Prove the following statements. Also draw a picture for the case $V = \mathbb{R}^2$.
    (a) $(x - y) \perp (x + y)$ iff $\|x\| = \|y\|$.
    (b) $(x - z) \perp (y - z)$ iff $\|x - z\|^2 + \|y - z\|^2 = \|x - y\|^2$.
    (c) $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.
    (d) $|(x, y)| \le \|x\| \, \|y\|$.
    (e) $\|x + y\| \le \|x\| + \|y\|$.

62. Let V be a real vector space with inner product $(\cdot, \cdot)$. Let $\{e_i : \ i \in \mathbb{N}\} \subset V$ be orthonormal, i.e., (e_i, e_j) is zero if $i \neq j$ and is one if $i = j$. Let $n \in \mathbb{N}$, $x \in V$, and $\lambda_i \in \mathbb{R}$ for $i \in \mathbb{N}$. Prove the following:
    (a) $\left\| \sum_{i=1}^n \lambda_i e_i \right\|^2 = \sum_{i=1}^n |\lambda_i|^2$;
    (b) $\left\| x - \sum_{i=1}^n \lambda_i e_i \right\|^2 = \|x\|^2 + \sum_{i=1}^n |\lambda_i - c_i|^2 - \sum_{i=1}^n |c_i|^2$ with $c_i = (x, e_i)$;
    (c) $\lim_{n\to\infty} \sum_{i=1}^n |(x, e_i)|^2$ exists and is less than or equal to $\|x\|^2$.

63. For this problem, use the inner products defined earlier for the various cases, respectively.
    (a) Find a set of three orthonormal vectors in $\mathbb{R}^3$.
    (b) Find α such that $e_n(t) = \alpha \mathrm{e}^{\mathrm{i}nt}$, $n \in \mathbb{Z}$ are orthonormal on $[-\pi, \pi]$.
    (c) For the set of real-valued polynomials on $[-1, 1]$, show that $p(x) = x$ is orthogonal to every constant function. Next, find a quadratic polynomial that is orthogonal to both p and the constant functions. Finally, find a cubic polynomial that is orthogonal to all quadratic polynomials. Hence construct an orthonormal set with three vectors.