Missouri University of Science and Technology

Section 4.3

Auxiliary Equations with Complex Roots

Recall: Second-Order Linear Equations

A second-order linear ODE is an ODE which can be written in the form

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

If $f(t) = 0$, the equation is called homogeneous.

Recall: The Auxiliary Equation

When solving the DE $ay'' + by' + cy = 0$, we assume $y = e^{rt}$ and obtain the auxiliary equation

$$ar^2 + br + c = 0$$

When solving the auxiliary equation, we will encounter one of these three cases:

I. Two real, distinct roots
II. One real, repeated root
III. Two complex roots (which occur in a conjugate pair)

Complex Roots of the Auxiliary Equation

If we find the complex roots

$$r = \lambda \pm \mu i$$

then we obtain two complex solutions to the DE:

$$z_1 = e^{(\lambda+\mu i)t} \text{ and } z_2 = e^{(\lambda-\mu i)t}$$

Unfortunately, we don't want complex solutions.
We want real solutions!

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Real Solutions from Complex Solutions

$$\begin{aligned} z_1 &= e^{(\lambda+\mu i)t} \\ &= e^{\lambda t}e^{\mu i t} \\ &= e^{\lambda t}[\cos(\mu t) + i\sin(\mu t)] \\ &= e^{\lambda t}\cos(\mu t) + ie^{\lambda t}\sin(\mu t) \end{aligned}$$

For simplicity, let $y_1 = e^{\lambda t}\cos(\mu t)$ and $y_2 = e^{\lambda t}\sin(\mu t)$.

Thus, $z_1 = y_1 + iy_2$ is a solution of the DE $ay'' + by' + cy = 0$.

Real Solutions from Complex Solutions

$z_1 = y_1 + iy_2$ is a solution of the DE $ay'' + by' + cy = 0$.

$$z_1' = y_1' + iy_2'$$
$$z_1'' = y_1'' + iy_2''$$

Substituting into our DE, we get

$$a(y_1'' + iy_2'') + b(y_1' + iy_2') + c(y_1 + iy_2) = 0$$

and we regroup the terms to get

$$(ay_1'' + by_1' + cy_1) + i(ay_2'' + by_2' + cy_2) = 0$$

Real Solutions from Complex Solutions

$$(ay_1'' + by_1' + cy_1) + i(ay_2'' + by_2' + cy_2) = 0$$

The only way this can be a true statement (which we know it is) is if both the real and imaginary parts are zero.

Thus,

$$ay_1'' + by_1' + cy_1 = 0$$

and

$$ay_2'' + by_2' + cy_2 = 0$$

meaning y_1 and y_2 are both real solutions of the DE.

Complex Roots of the Auxiliary Equation

Consider the differential equation $ay'' + by' + cy = 0$.

If its auxiliary equation $ar^2 + br + c = 0$ has complex roots $r = \lambda \pm \mu i$, then

$$y_1 = e^{\lambda t}\cos(\mu t)$$

and

$$y_2 = e^{\lambda t}\sin(\mu t)$$

are linearly independent real solutions of the DE.

Example 1

Find the general solution of the differential equation

$$y'' + 2y' + 2y = 0$$

Example 2

Find the general solution of the differential equation

$$y'' + y' + y = 0$$