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Sections 4.4 and 4.5

Nonhomogeneous Equations:
The Method of Undetermined Coefficients

Nonhomogeneous Equations

Our goal in these sections is to consider nonhomogeneous linear differential equations of the form

$$ay'' + by' + cy = g(t)$$

for certain particularly nice functions $g(t)$.

We will wait until the next section to consider non-constant coefficients and less desirable functions $g(t)$.

Superposition Principle for Nonhomogeneous Linear DEs

Consider the equation

$$ay'' + by' + cy = g(t)$$

If $y_h(t)$ is the general solution of the associated homogeneous equation

$$ay'' + by' + cy = 0$$

and $y_p(t)$ is any particular solution of the nonhomogeneous DE, then the general solution of the nonhomogeneous DE is

$$y = y_h(t) + y_p(t)$$

Homogeneous and Particular Solutions

The homogeneous solution $y_h(t)$ will always include at least one arbitrary constant.

The particular solution $y_p(t)$ will never contain an arbitrary constant.

Some sources refer to $y_h(t)$ as the complementary solution, often denoted $y_c(t)$.

Linear Combinations

A linear combination of two functions $f_1(t)$ and $f_2(t)$ is

$$c_1 f_1(t) + c_2 f_2(t)$$

where c_1 and c_2 are constants.

Linear combinations of three or more functions are defined similarly.

The Method of Undetermined Coefficients

When solving $ay'' + by' + cy = g(t)$, the Method of Undetermined Coefficients is a method in which we use the form of $g(t)$ to guess the form of the particular solution $y_p(t)$.

Specifically, the form of $y_p(t)$ is a linear combination of $g(t)$ and all distinct functions generated by repeated differentiation of $g(t)$, provided that none of those functions duplicate the functions found in the homogeneous solution $y_h(t)$.

Example 1

Solve the equation

$$2y'' + 3y' + y = t^2$$

given that the homogeneous solution for this equation is

$$y_h(t) = C_1 e^{-t/2} + C_2 e^{-t}$$

Example 2 – Form of $y_p(t)$

For each given function $g(t)$ from an equation of the form $ay'' + by' + cy = g(t)$, identify the form of $y_p(t)$.

Example 2 – Form of $y_p(t)$

$g(t)$	**Form of $y_p(t)$**
Some constant	A
An nth degree polynomial	An nth degree polynomial
$\sin qt$ or $\cos qt$	$A \sin qt + B \cos qt$
e^{pt}	Ae^{pt}
$e^{pt} \sin qt$ or $e^{pt} \cos qt$	$Ae^{pt} \sin qt + Be^{pt} \cos qt$
$4t \sin 3t$	$(At + B) \sin 3t + (Ct + E) \cos 3t$
$t^3 e^{7t}$	$(At^3 + Bt^2 + Ct + E)e^{7t}$

Example 3

Solve the equation

$$2y'' + 3y' + y = e^{3t}$$

given that the homogeneous solution for this equation is

$$y_h(t) = C_1 e^{-t/2} + C_2 e^{-t}$$

Undetermined Coefficients – When y_p Duplicates y_h

If our initial guess for y_p duplicates y_h, multiply the entire guess for y_p by the smallest positive integer power of t which ensures that no portion of y_h is duplicated.

Example 4

Solve the equation

$$2y'' + 3y' + y = e^{-t}$$

given that the homogeneous solution for this equation is

$$y_h(t) = C_1 e^{-t/2} + C_2 e^{-t}$$

Example 5

Solve the equation

$$2y'' + 3y' + y = t^2 + e^{-t}$$

given that the homogeneous solution for this equation is

$$y_h(t) = C_1 e^{-t/2} + C_2 e^{-t}$$

Example 6 (if time permits)

Solve the equation

$$3y'' - y' - 14y = e^{3t}$$

Example 7 (if time permits)

Solve the equation

$$3y'' - y' - 14y = 7e^{-2t}$$