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Dynamic equations on time scales: a survey

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Abstract

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger (1988), is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. In this paper we give an introduction to the time scales calculus. We also present various properties of the exponential function on an arbitrary time scale, and use it to solve linear dynamic equations of first order. Several examples and applications, among them an insect population model, are considered. We then use the exponential function to define hyperbolic and trigonometric functions and use those to solve linear dynamic equations of second order with constant coefficients. Finally, we consider self-adjoint equations and, more generally, so-called symplectic systems, and present several results on the positivity of quadratic functionals. © 2002 Elsevier Science B.V. All rights reserved.

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1. Unifying continuous and discrete analysis

In 1988, Stefan Hilger [12] introduced the calculus of *measure chains* in order to unify continuous and discrete analysis. Bernd Aulbach, who supervised Stefan Hilger's Ph.D. thesis [11], points out the three main purposes of this new calculus:

Unification – Extension – Discretization.

For many purposes in analysis it is sufficient to consider a special case of a measure chain, a so-called *time scale*, which simply is a closed subset of the real numbers. We denote a time scale

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by the symbol $\mathbb{T}$. For functions y defined on $\mathbb{T}$, we can consider a so-called delta derivative y^{Δ}, and this delta derivative is equal to y' (the usual derivative) if $\mathbb{T}=\mathbb{R}$ is the set of all real numbers, and it is equal to Δy (the usual forward difference) if $\mathbb{T}=\mathbb{Z}$ is the set of all integers. Then we can study *dynamic equations*

$$f(t, y, y^{\Delta}, y^{\Delta^2}, \ldots, y^{\Delta^n}) = 0,$$

which may also involve higher order derivatives as indicated. Along with such dynamic equations we can consider initial value problems and boundary value problems. We remark that these dynamic equations are differential equations when $\mathbb{T}=\mathbb{R}$, and they are difference equations when $\mathbb{T}=\mathbb{Z}$. Other kinds of equations are covered by them as well, such as for example *q-difference equations*

$$\mathbb{T} = q^{\mathbb{Z}} := \{q^k \mid k \in \mathbb{Z}\} \cup \{0\} \quad \text{for some } q > 1$$

or difference equations with constant step size

$$\mathbb{T} = h\mathbb{Z} := \{hk \mid k \in \mathbb{Z}\} \quad \text{for some } h > 0.$$

Particularly useful for the discretization aspect are time scales of the form

$$\mathbb{T} = \{t_k \mid k \in \mathbb{Z}\} \quad \text{where } t_k \in \mathbb{R},\ t_k < t_{k+1} \quad \text{for all } k \in \mathbb{Z}.$$

This survey paper is organized as follows: In Section 2 we introduce the basic concepts of the time scales calculus. One major object, when studying dynamic equations on time scales, is the *exponential function*, which will be discussed in Section 3. Section 4 is devoted to examples and applications. In Section 5 we consider linear dynamic equations and initial value problems involving them. Finally, in Section 6, we study a particular case of linear dynamic systems, namely symplectic systems on time scales. The bibliography at the end of this paper contains, besides references that are cited in the text, many of the recent articles in this relatively new area of research. The book [7] is an up-to-date summary on the subject.

2. Delta derivatives

A time scale is a nonempty closed subset of the reals, and we usually denote it by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$. We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} \mid s < t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). A point $t \in \mathbb{T}$ is called *right-scattered*, *right-dense*, *left-scattered*, *left-dense*, if $\sigma(t) > t,\ \sigma(t) = t,\ \rho(t) < t,\ \rho(t) = t$ holds, respectively. The set $\mathbb{T}^{\kappa}$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum. The *graininess* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

Hence the graininess function is constant 0 if $\mathbb{T}=\mathbb{R}$ while it is constant 1 for $\mathbb{T}=\mathbb{Z}$. However, a time scale $\mathbb{T}$ could have nonconstant graininess. Now, let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* (or simply: *differentiable*) at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\varepsilon > 0$ there is a neighborhood $\mathscr{N}$ around t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for all } s \in \mathscr{N}.$$

In this case we denote the α by $f^\Delta(t)$, and if f is differentiable for every $t \in \mathbb{T}^\kappa$, then f is said to be *differentiable* on $\mathbb{T}$ and f^Δ is a new function defined on $\mathbb{T}^\kappa$. If f is differentiable at $t \in \mathbb{T}^\kappa$, then it is easy to see that

$$f^\Delta(t) = \begin{cases} \lim_{s \to t, s \in \mathbb{T}} \dfrac{f(t) - f(s)}{t - s} & \text{if } \mu(t) = 0 \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)} & \text{if } \mu(t) > 0. \end{cases} \tag{2.1}$$

However, it is exactly the philosophy of the calculus on time scales to avoid the separate discussion of the two cases $\mu(t) = 0$ and $\mu(t) > 0$, i.e., when t is right-dense and right-scattered, respectively. Results as in (2.1) do not serve this purpose and therefore should be avoided in proofs. To illustrate the idea, we now give another formula, which holds whenever f is differentiable at $t \in \mathbb{T}^\kappa$:

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{2.2}$$

When applying formula (2.2), we do not need to distinguish between the two cases $\mu(t) = 0$ and $\mu(t) > 0$. Formula (2.2) holds in both of these cases. Even though it is trivial in the case of a right-dense t, we need not worry about different cases. Two further examples of such formulas are the product rule for the derivative of the product of two differentiable functions f and g:

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f(\sigma(t)) g^\Delta(t) \tag{2.3}$$

and the quotient rule for the derivative of the quotient of two differentiable functions f and $g \neq 0$:

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t) g(t) - f(t) g^\Delta(t)}{g(t) g(\sigma(t))}. \tag{2.4}$$

Clearly, $1^\Delta = 0$ and $t^\Delta = 1$, so we can use (2.3) to find

$$(t^2)^\Delta = (t \cdot t)^\Delta = t + \sigma(t),$$

and we can use (2.4) to find

$$\left(\frac{1}{t}\right)^\Delta = -\frac{1}{t\sigma(t)}.$$

Other formulas may be obtained likewise. One word of caution: The forward jump operator σ is not necessarily a differentiable function. Clearly, at a point which is left-dense and right-scattered at the same time, σ is not continuous. Hence σ is not differentiable at such a point since we know:

Theorem 2.1. *Every differentiable function is continuous.*

However, σ is an example of a function which we call rd-continuous. A function f defined on $\mathbb{T}$ is *rd-continuous*, if it is continuous at every right-dense point and if the left-sided limit exists in every left-dense point. The importance of rd-continuous functions is revealed by the following existence result by Hilger [12]:

Theorem 2.2. *Every rd-continuous function possesses an antiderivative.*

Here, F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^\Delta = f$ holds on $\mathbb{T}^\kappa$. In this case we define an integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \quad \text{where } s, t \in \mathbb{T}.$$

An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a polynomial (or any "nice" formula of a function) which is an antiderivative of t (where $\mathbb{T}$ is an arbitrary time scale). The role of t^2 is therefore played in the time scales calculus by

$$\int_0^t \sigma(\tau)\Delta\tau \quad \text{and} \quad \int_0^t \tau\Delta\tau.$$

Note that both integrals exist by Theorem 2.2 as the functions σ and identity are both rd-continuous. In general, the functions

$$g_0(t, s) \equiv 1 \quad \text{and} \quad g_{k+1}(t, s) = \int_s^t g_k(\sigma(\tau), s)\Delta\tau, \quad k \geqslant 0$$

and

$$h_0(t, s) \equiv 1 \quad \text{and} \quad h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau, \quad k \geqslant 0$$

may be considered as the "polynomials" on $\mathbb{T}$. The relationship between g_k and h_k is given in [2] as

$$g_k(t, s) = (-1)^k h_k(t, s) \quad \text{for all } k \in \mathbb{N}. \tag{2.5}$$

To verify (2.5) e.g., for the case $k = 2$ we can calculate

$$\begin{aligned} g_2(t, s) &= \int_s^t g_1(\sigma(\tau), s)\Delta\tau = \int_s^t (\sigma(\tau) - s)\Delta\tau \\ &= \int_s^t (\sigma(\tau) + \tau)\Delta\tau - \int_s^t \tau\Delta\tau - \int_s^t s\Delta\tau \\ &= \int_s^t (\tau^2)^\Delta\Delta\tau + \int_t^s \tau\Delta\tau - s(t - s) \\ &= \int_t^s \tau\Delta\tau + t^2 - s^2 - s(t - s) \\ &= \int_t^s (\tau - t)\Delta\tau = \int_t^s h_1(\tau, t)\Delta\tau = h_2(s, t). \end{aligned}$$

One of the two versions of Taylor's formula (the other obtained by using (2.5)) reads as follows, see [2]:

Theorem 2.3. *Let* $\alpha \in \mathbb{T}^{\kappa^{n-1}}$. *If* f *is* n *times differentiable, then*

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

3. Special functions

A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

Concerning initial value problems

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1 \tag{3.1}$$

(where $t_0 \in \mathbb{T}$, as is assumed throughout this paper), Hilger [12] proved the following existence and uniqueness theorem:

Theorem 3.1. *If* p *is rd-continuous and regressive, then* (3.1) *has a unique solution.*

We call the unique solution of (3.1) the *exponential function* and denote it by $\mathrm{e}_p(\cdot, t_0)$. In fact, there is an explicit formula for $\mathrm{e}_p(t, s)$, using the so-called cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1+hz)}{z} & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}$$

The formula, see [13], reads

$$\mathrm{e}_p(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\}. \tag{3.2}$$

It can be seen immediately from (3.2) that the exponential function never vanishes; however, in contrast to the case $\mathbb{T} = \mathbb{R}$, the exponential function could possibly attain negative values. As an example, just consider the problem $y^{\Delta} = -2y$, $y(0) = 1$ for $\mathbb{T} = \mathbb{Z}$. Now we will mention some properties of the exponential function. First, we have by (2.2)

$$\begin{aligned} \mathrm{e}_p(\sigma(t), s) &= \mathrm{e}_p(t, s) + \mu(t)\mathrm{e}_p^{\Delta}(t, s) \\ &= \mathrm{e}_p(t, s) + \mu(t)p(t)\mathrm{e}_p(t, s) \\ &= [1 + \mu(t)p(t)]\mathrm{e}_p(t, s), \end{aligned}$$

where we denote by $e_p^\Delta(t, s)$ the derivative of e_p with respect to the first variable. Using this equation and letting $y = e_p(\cdot, t_0)e_q(\cdot, t_0)$ for rd-continuous and regressive p and q, we find by (2.3) that

$$\begin{aligned} y^\Delta(t) &= e_p^\Delta(t, t_0)e_q(t, t_0) + e_p(\sigma(t), t_0)e_q^\Delta(t, t_0) \\ &= p(t)e_p(t, t_0)e_q(t, t_0) + [1 + \mu(t)p(t)]e_p(t, t_0)q(t)e_q(t, t_0) \\ &= [p(t) + q(t) + \mu(t)p(t)q(t)]y(t). \end{aligned}$$

Hence, introducing an addition $\oplus$ by

$$p \oplus q = p + q + \mu pq,$$

we find that y solves the initial value problem

$$y^\Delta = (p \oplus q)(t)y, \quad y(t_0) = 1,$$

and therefore $y = e_{p\oplus q}(\cdot, t_0)$ by Theorem 3.1. Thus

$$e_{p\oplus q} = e_p \cdot e_q.$$

Note also that $p \oplus q$ is regressive iff both p and q are:

$$1 + \mu(p \oplus q) = 1 + \mu(p + q + \mu pq) = 1 + \mu p + \mu q + \mu^2 pq = (1 + \mu p)(1 + \mu q).$$

To see how the corresponding subtraction $\ominus$ should be defined, we denote $y = e_p(\cdot, t_0)/e_q(\cdot, t_0)$ and use (2.4) to calculate

$$\begin{aligned} y^\Delta(t) &= \frac{e_p^\Delta(t, t_0)e_q(t, t_0) - e_p(t, t_0)e_q^\Delta(t, t_0)}{e_q(t, t_0)e_q(\sigma(t), t_0)} \\ &= \frac{p(t)e_p(t, t_0)e_q(t, t_0) - e_p(t, t_0)q(t)e_q(t, t_0)}{e_q(t, t_0)[1 + \mu(t)q(t)]e_q(t, t_0)} \\ &= \frac{p(t) - q(t)}{1 + \mu(t)q(t)}y(t) \end{aligned}$$

so that y solves the initial value problem

$$y^\Delta = (p \ominus q)(t)y, \quad y(t_0) = 1$$

provided we introduce the subtraction $\ominus$ by

$$p \ominus q = \frac{p - q}{1 + \mu q}.$$

Therefore, again by Theorem 3.1,

$$e_{p\ominus q} = \frac{e_p}{e_q}.$$

Note again that $p \ominus q$ is regressive if both p and q are:

$$1 + \mu(p \ominus q) = 1 + \mu \frac{p-q}{1+\mu q} = \frac{1 + \mu q + \mu(p-q)}{1+\mu q} = \frac{1+\mu p}{1+\mu q}.$$

We put

$$\ominus q = 0 \ominus q = -\frac{q}{1+\mu q},$$

and then we can derive many useful formulas such as e.g., $\ominus(\ominus q) = q$. We have

Theorem 3.2. *The set of regressive functions together with addition $\oplus$ is an Abelian group.*

Further properties of the exponential function may be obtained as in [8,13]. Next we want to emphasize, that, along with the equation $y^\Delta = p(t)y$, there is another "natural" form of a linear equation of first order, namely $x^\Delta = -p(t)x^\sigma$, where we denote $x^\sigma = x \circ \sigma$. Of course for $\mathbb{T} = \mathbb{R}$ both these equations are (up to a minus sign) the same. However, in the general time scales setting, the two equations are different. Let us now solve the initial value problem

$$x^\Delta = -p(t)x^\sigma, \quad x(t_0) = 1, \tag{3.3}$$

subject to the usual assumptions on p. Assuming that x solves (3.3), we find by using (2.2)

$$x^\Delta(t) = -p(t)x(\sigma(t)) = -p(t)[x(t) + \mu(t)x^\Delta(t)]$$

and hence

$$[1 + \mu(t)p(t)]x^\Delta(t) = -p(t)x(t)$$

so that (note that p is assumed to be regressive)

$$x^\Delta(t) = -\frac{p(t)}{1+\mu(t)p(t)}x(t) = (\ominus p)(t)x(t).$$

Therefore x is a solution of

$$y^\Delta = (\ominus p)(t)y, \quad y(t_0) = 1.$$

Theorem 3.3. *If p is rd-continuous and regressive, then $e_p(\cdot, t_0)$ and $e_{\ominus p}(\cdot, t_0)$ are the unique solutions of* (3.1) *and* (3.3), *respectively.*

Using the exponential function, we now may find solutions of many dynamic equations in a similar way as we would do it with ordinary differential equations. As an example, we consider the equation

$$y^{\Delta^3} - 2y^{\Delta^2} - y^\Delta + 2y = 0.$$

Trying $y(t)=e_\lambda(t,t_0)$ with constant λ leads us to

$$0=(\lambda^3-2\lambda^2-\lambda+2)e_\lambda(t,t_0)=(\lambda+1)(\lambda-1)(\lambda-2)e_\lambda(t,t_0),$$

i.e., $\lambda\in\{-1,1,2\}$ so that $e_{-1}(\cdot,t_0)$, $e_1(\cdot,t_0)$, and $e_2(\cdot,t_0)$ are solutions. All solutions of this dynamic equation may be constructed by taking linear combinations of the above three solutions. As another example, we consider the initial value problem

$$y^{\Delta^2}=a^2y,\quad y(t_0)=1,\quad y^\Delta(t_0)=0, \tag{3.4}$$

where a is a regressive constant. By using the technique described above, we find that both $e_a(\cdot,t_0)$ and $e_{-a}(t,t_0)$ solve the dynamic equation. Therefore we try a linear combination of the form $\alpha e_a(\cdot,t_0)+\beta e_{-a}(\cdot,t_0)$. We find $\alpha+\beta=1$ and $\alpha-\beta=0$ so that $\alpha=\beta=1/2$. Hence it is useful to introduce

$$\cosh_p=\frac{e_p+e_{-p}}{2}\quad\text{and}\quad\sinh_p=\frac{e_p-e_{-p}}{2},$$

where sinh is the derivative of cosh. Both definitions require p and $-p$ to be regressive (and rd-continuous), and because of

$$(1+\mu p)(1-\mu p)=1-\mu^2p^2,$$

this is equivalent to $-\mu p^2$ being regressive. The following result is easy to verify:

Theorem 3.4. *If p is rd-continuous and $-\mu p^2$ is regressive, then*

$$\sinh_p^\Delta=\cosh_p,\quad\cosh_p^\Delta=\sinh_p,\quad\textit{and}\quad\cosh_p^2-\sinh_p^2=e_{-\mu p^2}.$$

Now we also introduce trigonometric functions in a similar fashion:

$$\cos_p=\frac{e_{ip}+e_{-ip}}{2}\quad\text{and}\quad\sin_p=\frac{e_{ip}-e_{-ip}}{2i},$$

provided μp^2 is regressive, and this is clearly satisfied if $p(t)\in\mathbb{R}$ for all $t\in\mathbb{T}$. We find the following result:

Theorem 3.5. *If p is rd-continuous and real, then*

$$\sin_p^\Delta=\cos_p,\quad\cos_p^\Delta=-\sin_p,\quad\textit{and}\quad\sin_p^2+\cos_p^2=e_{\mu p^2}.$$

In particular, if $a\in\mathbb{R}$ is constant, then $\cos_a(\cdot,t_0)$ solves the initial value problem

$$y^{\Delta^2}=-ay,\quad y(t_0)=1,\quad y^\Delta(t_0)=0.$$

Finally, we mention the analogue of Theorem 3.1 in the matrix case. This result can also be used to study higher order equations by rewriting them first in matrix form. Let P be an $n\times n$-matrix-valued, rd-continuous (i.e., each entry is rd-continuous) function of $\mathbb{T}$. We say that P is *regressive* provided

$$I+\mu(t)P(t)\quad\text{is invertible for all } t\in\mathbb{T},$$

Table 1
The two most important examples

Time scale $\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
Backward jump operator $\rho(t)$	t	$t-1$
Forward jump operator $\sigma(t)$	t	$t+1$
Graininess $\mu(t)$	0	1
Derivative $f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$
Integral $\int_a^b f(t)\Delta t$	$\int_a^b f(t)\,\mathrm{d}t$	$\sum_{t=a}^{b-1} f(t)$ (if $a<b$)
Rd-continuous f	continuous f	any f
κ-operator $\mathscr{I}^{\kappa}$	$\mathscr{I}$	$\mathscr{I}\backslash\{\max\mathscr{I}\}$

where I denotes the $n\times n$-identity matrix. Concerning initial value problems of the form

$$Y^{\Delta}=P(t)Y, \quad Y(t_0)=I \tag{3.5}$$

we have the following result of Hilger [12]:

Theorem 3.6. *If P is rd-continuous and regressive, then* (3.5) *has a unique solution.*

We refer to this solution as the *matrix exponential* and denote it by $\mathrm{e}_P(\cdot,t_0)$. In the case of a constant matrix A it is possible to calculate $\mathrm{e}_A(\cdot,t_0)$ using a generalization of Putzer's algorithm as is explained in [3].

4. Examples and applications

Example 4.1. Of course the two most popular examples of time scales are $\mathbb{R}$ and $\mathbb{Z}$. In Table 1 we collect some information about these time scales.

Now let us solve (3.1) for these two cases and hence find the corresponding exponential functions: Clearly, if $\mathbb{T}=\mathbb{R}$, then

$$\mathrm{e}_p(t,s)=\exp\left\{\int_s^t p(\tau)\,\mathrm{d}\tau\right\} \quad \text{and} \quad \mathrm{e}_a(t,s)=e^{a(t-s)}$$

if p is continuous and a is constant. Hence e.g.,

$$\mathrm{e}_a(t,0)=e^{at} \quad \text{and} \quad \mathrm{e}_1(t,0)=e^t.$$

Similarly,

$$\sinh_p(t,s)=\sinh\left\{\int_s^t p(\tau)\,\mathrm{d}\tau\right\}, \quad \sinh_a(t,s)=\sinh(a(t-s)),$$

and hence e.g.,

$$\sinh_a(t,0)=\sinh(at) \quad \text{and} \quad \sinh_1(t,0)=\sinh t,$$

and corresponding formulas can be written down for cosh, sin, and cos. The functions h_k and g_k are given by

$$h_k(t,s)=g_k(t,s)=\frac{(t-s)^k}{k!},\quad \text{e.g., } h_k(t,0)=g_k(t,0)=\frac{t^k}{k!}.$$

Next, if $\mathbb{T}=\mathbb{Z}$, then (3.1) reads

$$\Delta y(t):=y(t+1)-y(t)=p(t)y(t+1),\quad y(t_0)=1.$$

Hence, if $p(t)\neq -1$ for all $t\in\mathbb{Z}$, then

$$\mathrm{e}_p(t,s)=\begin{cases}\prod_{\tau=s}^{t-1}[1+p(\tau)] & \text{if} \quad t>s\\ 1 & \text{if} \quad t=s\\ \dfrac{1}{\prod_{\tau=t}^{s-1}[1+p(\tau)]} & \text{if} \quad t<s\end{cases}$$

and therefore with constant $a\neq -1$

$$\mathrm{e}_a(t,s)=(1+a)^{t-s},\quad \mathrm{e}_a(t,0)=(1+a)^t,\quad \mathrm{e}_1(t,0)=2^t.$$

Moreover, e.g.,

$$\cosh_a(t,0)=\frac{(1+a)^t+(1-a)^t}{2}=\sum_{k=0}^{\infty}\binom{t}{2k+1}a^{2k+1}$$

and

$$\sinh_a(t,0)=\frac{(1+a)^t-(1-a)^t}{2}=\sum_{k=0}^{\infty}\binom{t}{2k}a^{2k}$$

and similarly for the trigonometric functions. The functions h_k and g_k are given by

$$h_k(t,s)=\binom{t-s}{k}\quad\text{and}\quad g_k(t,s)=\binom{t-s+k-1}{k}\quad\text{if } t>s.$$

Example 4.2. Another important time scale is

$$h\mathbb{Z}=\{hk\mid k\in\mathbb{Z}\}\quad\text{for some } h>0.$$

Dynamic equations on $h\mathbb{Z}$ correspond to difference equations with step size h rather than 1. Hence (3.1) reads

$$\frac{y(t+h)-y(t)}{h}=p(t)y(t),\quad y(t_0)=1.$$

E.g., if $h=1/n$ for some natural number $n\in\mathbb{N}$, then

$$\mathrm{e}_a(t,0)=\left(1+\frac{a}{n}\right)^{nt}\to e^{at}\quad\text{as } n\to\infty.$$

Many properties on this time scale "become" the corresponding "continuous" properties as $h \to 0$. As one more example we use Theorem 3.5 to find

$$\sin_a^2(t,0) + \cos_a^2(t,0) = \mathrm{e}_{a^2/n}(t,0) = \left(1 + \frac{a^2}{n^2}\right)^{nt} \to 1 \quad \text{as } n \to \infty.$$

The use of $h\mathbb{Z}$ lies in the "discretization" aspect of the time scales calculus. Time scales which serve this purpose even better are of the form

$$\mathbb{T} = \{t_k(h) \mid k \in \mathbb{Z}\} \quad \text{with } 0 \leqslant t_{k+1}(h) - t_k(h) = o(h) \text{ as } h \to 0.$$

They allow discretization with variable step size.

Example 4.3. Let $N(t)$ be the amount of plants of one particular kind at time t in a certain area. By experiments we know that N grows exponentially according to $N' = N$ during the months of April until September. At the beginning of October, all plants suddenly die, but the seeds remain in the ground and start growing again at the beginning of April with N now being doubled. We model this situation using the time scale

$$\mathbb{T} = \bigcup_{k=0}^{\infty} [2k, 2k+1],$$

where $t = 0$ is April 1 of the current year, $t = 1$ is October 1 of the current year, $t = 2$ is April 1 of the next year, $t = 3$ is October 1 of the next year, and so on. We have

$$\mu(t) = \begin{cases} 0 & \text{if } 2k \leqslant t < 2k+1 \\ 1 & \text{if } t = 2k+1. \end{cases}$$

On $[2k, 2k+1)$, we have $N' = N$, i.e., $N^\Delta = N$. However, we also know that $N(2k+2) = 2N(2k+1)$, i.e., $\Delta N(2k+1) = N(2k+1)$, i.e., $N^\Delta = N$ at $2k+1$. As a result, N is a solution of the dynamic equation

$$N^\Delta = N.$$

Thus, if $N(0) = 1$ is given, N is exactly $\mathrm{e}_1(\cdot, 0)$ on the time scale $\mathbb{T}$. Further examples that can be modeled with similar time scales include insect population models, which are discrete in season (and may follow a difference scheme with variable step size or are often modeled by continuous dynamic systems), die out in say winter, while their eggs are incubating or dormant, and then in season again, hatching gives rise to a nonoverlapping population. Some specific examples [9] are the cicada *Magicicada septendecim* which lives as a larva for 17 years and as an adult for perhaps a week, and the common mayfly *Stenonema canadense* which lives as a larva for a year and as an adult for less than a day.

Example 4.4 (S. Keller [15]). Consider a simple electric circuit with resistance R, inductance L, and capacitance C. Suppose we decharge the capacitor periodically every time unit and assume that the decharging takes $\delta > 0$ (but small) time units. Then this stimulation can be modeled using the time scale $\mathbb{T} = \bigcup_{k \in \mathbb{N}_0} [k, k+1-\delta]$. If $Q(t)$ is the total charge on the capacitor at time t and $I(t)$ is the current as a function of time t, then the corresponding dynamic system that is satisfied by Q and I (on $\mathbb{T}$) reads as follows:

$$Q^\Delta = \begin{cases} bQ & \text{on } \bigcup_{k\in\mathbb{N}}\{k-\delta\} \\ I & \text{otherwise,} \end{cases} \quad \text{and} \quad I^\Delta = \begin{cases} 0 & \text{on } \bigcup_{k\in\mathbb{N}}\{k-\delta\} \\ -\frac{1}{LC}Q - \frac{R}{L}I & \text{otherwise.} \end{cases}$$

Example 4.5. So-called *q-difference equations*, $q > 1$ have been studied extensively in the literature, see e.g., [4,16,17]. An example of such an equation is

$$y(qt) - y(t) = (q-1)t.$$

The time scale, which has such q-difference equations as its dynamic equations, is given by

$$q^{\mathbb{N}_0} = \{q^k \mid k \in \mathbb{N}_0\} \quad \text{for some } q > 1,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We have $\sigma(t) = qt$ for all $t \in q^{\mathbb{N}_0}$. If $t_0 = 1$, then (3.1) reads

$$\frac{y(qt) - y(t)}{(q-1)t} = p(t)y(t), \quad y(1) = 1.$$

This yields

$$\mathrm{e}_p(q^k, 1) = \prod_{v=0}^{k-1} [1 + (q-1)q^v p(q^v)] \quad \text{for all } k \in \mathbb{N}.$$

In particular, if $p(s) = (1-s)/(q-1)s^2$, then for $t = q^k$

$$\begin{aligned} \mathrm{e}_1(t,1) = \mathrm{e}_1(q^k,1) &= \prod_{v=0}^{k-1}\left[1 + \frac{1-q^v}{q^v}\right] = \prod_{v=0}^{k-1}\frac{1}{q^v} = \frac{1}{q^{k(k-1)/2}} \\ &= q^{-k^2/2}q^{k/2} = \sqrt{q^k}\exp\left\{-\frac{k^2\log q}{2}\right\} = \sqrt{q^k}\exp\left\{-\frac{(k\log q)^2}{2\log q}\right\} \\ &= \sqrt{t}\exp\left\{-\frac{\log^2 t}{2\log q}\right\}. \end{aligned}$$

Moreover, in [2] we find

$$h_k(t,s) = \prod_{v=0}^{k-1}\frac{t - q^v s}{\sum_{\mu=0}^{v} q^\mu} \quad \text{if } t > s.$$

E.g., if $q = 2$, we have

$$h_2(t,1) = \frac{(t-1)(t-2)}{3}, \quad h_3(t,1) = \frac{(t-1)(t-2)(t-4)}{21},$$

and

$$h_4(t,1) = \frac{(t-1)(t-2)(t-4)(t-8)}{315}.$$

Example 4.6. More generally, let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers and let $t_0 \in \mathbb{R}$. Let

$$t_k = t_0 + \sum_{v=1}^{k} \alpha_v \quad \text{for } k \in \mathbb{N}.$$

Put

$$\mathbb{T} = \begin{cases} \{t_k \mid k \in \mathbb{N}_0\} & \text{if } \lim_{k\to\infty} t_k = \infty \\ \{t_k \mid k \in \mathbb{N}_0\} \cup \{t^*\} & \text{if } \lim_{k\to\infty} t_k = t^*. \end{cases}$$

Then

$$\sigma(t_k) = t_{k+1} \quad \text{and} \quad \mu(t_k) = \alpha_{k+1} \quad \text{for all } k \in \mathbb{N}_0.$$

Problem (3.1) now reads

$$\frac{y(t_{k+1}) - y(t_k)}{\alpha_{k+1}} = p(t_k)y(t_k), \quad y(t_0) = 1,$$

and hence has the solution

$$e_p(t_k, t_0) = \prod_{v=1}^{k} [1 + \alpha_v p(t_{v-1})] \quad \text{for all } k \in \mathbb{N}_0. \tag{4.1}$$

Now we consider some more specific examples:

(i) Let $t_0 = 1$ and $\alpha_k = (1 - 1/q)q^k$ for $k \in \mathbb{N}$. Then

$$t_k = t_0 + \sum_{v=1}^{k} \alpha_v = 1 + \left(1 - \frac{1}{q}\right) \sum_{v=1}^{k} q^v = 1 + \frac{q-1}{q} \sum_{v=0}^{k-1} q^{v+1}$$

$$= 1 + (q-1) \sum_{v=0}^{k-1} q^v = 1 + (q-1)\frac{q^k - 1}{q - 1} = q^k,$$

so $\mathbb{T} = q^{\mathbb{N}_0}$. By (4.1), the solution of (3.1) is given by

$$e_p(q^k, 1) = \prod_{v=1}^{k} \left[1 + \left(1 - \frac{1}{q}\right) q^v p(q^{v-1})\right] = \prod_{v=0}^{k-1} [1 + (q-1)q^v p(q^v)],$$

compare with Example 4.5.

(ii) Let $t_0 = 0$ and $\alpha_k = 1/k$ for $k \in \mathbb{N}$. Then

$$\mathbb{T} = \left\{ \sum_{v=1}^{k} \frac{1}{v} \,\middle|\, k \in \mathbb{N} \right\} \cup 0.$$

Suppose $p(t) \equiv N \in \mathbb{N}$ is constant. By (4.1), the solution of (3.1) is given by

$$e_N(t_k, 0) = \prod_{v=1}^{k} \left(1 + \frac{N}{v}\right) = \prod_{v=1}^{k} \frac{N+v}{v} = \frac{(N+k)!}{N!k!} = \binom{N+k}{k}.$$

(iii) Let $t_0 = 1$ and $\alpha_k = 2k + 1$ for $k \in \mathbb{N}$. Then

$$t_k = t_0 + \sum_{v=1}^{k} \alpha_v = 1 + \sum_{v=1}^{k} (2v+1) = 1 + 2\frac{k(k+1)}{2} + k = (k+1)^2$$

and

$$\mathbb{T} = \mathbb{N}^2 := \{k^2 \mid k \in \mathbb{N}\}.$$

Suppose $p(t) \equiv 1$. By (4.1), the solution of (3.1) is given by

$$e_1((k+1)^2, 1) = e_1(t_k, 1) = \prod_{v=1}^{k} (1 + \alpha_v) = \prod_{v=1}^{k} (2 + 2v) = 2^k \prod_{v=1}^{k} (1 + v) = 2^k (k+1)!$$

so that

$$e_1(t, 1) = 2^{\sqrt{t}-1} (\sqrt{t})! \quad \text{for all } t \in \mathbb{T}.$$

(iv) Let $t_0 = 0$ and $\alpha_k = 1/(4k^2 - 1)$ for $k \in \mathbb{N}$. In this case

$$t_k = \sum_{v=1}^{k} \alpha_v = \sum_{v=1}^{k} \frac{1}{4v^2 - 1}$$

converges, say to t^*. Suppose $p(t) \equiv 1$. By (4.1),

$$e_1(t_k, 0) = \prod_{v=1}^{k} (1 + \alpha_v) = \prod_{v=1}^{k} \left(1 + \frac{1}{4v^2 - 1}\right) = \prod_{v=1}^{k} \frac{4v^2}{4v^2 - 1}$$

solves (3.1) and

$$\lim_{t \to t^*} e_1(t, 0) = \lim_{k \to \infty} \left\{ \prod_{v=1}^{k} \frac{4v^2}{4v^2 - 1} \right\} = \frac{\pi}{2}$$

by a well-known identity on this so-called *Wallis product*.

5. Linear dynamic equations

Clearly, using Theorem 3.3, the unique solution of

$$x^{\Delta} = -\, p(t) x^{\sigma}, \quad x(t_0) = x_0$$

is $x_0 e_{\ominus p}(\cdot, t_0)$, provided p is regressive and rd-continuous. If x solves

$$x^{\Delta} = -\, p(t) x^{\sigma} + f(t), \quad x(t_0) = x_0 \tag{5.1}$$

where, in addition, f is rd-continuous, then we multiply the dynamic equation in (5.1) by the *integrating factor* $\mathrm{e}_p(t,t_0)$ to obtain

$$[x\mathrm{e}_p(\cdot,t_0)]^\Delta = x^\Delta \mathrm{e}_p(\cdot,t_0) + p\mathrm{e}_p(\cdot,t_0)x^\sigma = \mathrm{e}_p(\cdot,t_0)[x^\Delta + px^\sigma] = \mathrm{e}_p(\cdot,t_0)f.$$

We integrate both sides between t_0 and t to arrive at a formula for x. Using (2.2), we also can solve the problem

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0. \tag{5.2}$$

Hence we obtain the following result from [8]:

Theorem 5.1. *If p and f are rd-continuous and p is regressive, then*

$$x_0\mathrm{e}_{\ominus p}(t,t_0) + \int_{t_0}^t \mathrm{e}_{\ominus p}(t,\tau)f(\tau)\Delta\tau \quad \textit{and} \quad y_0\mathrm{e}_p(t,t_0) + \int_{t_0}^t \mathrm{e}_p(t,\sigma(\tau))f(\tau)\Delta\tau$$

solve (5.1) *and* (5.2), *respectively.*

Now we consider second order equations with constant coefficients

$$y^{\Delta\Delta} + \alpha y^\Delta + \beta y = 0. \tag{5.3}$$

Trying $y = \mathrm{e}_\lambda(\cdot,t_0)$ with some regressive constant λ, we find that y solves (5.3) provided λ satisfies the *characteristic equation*

$$\lambda^2 + \alpha\lambda + \beta = 0,$$

which has solutions

$$\lambda_1 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \quad \text{and} \quad \lambda_2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}.$$

The *Wronskian* of $y_1 = \mathrm{e}_{\lambda_1}(\cdot,t_0)$ and $y_2 = \mathrm{e}_{\lambda_2}(\cdot,t_0)$ is defined as

$$W(y_1,y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1^\Delta & y_2^\Delta \end{pmatrix}$$

and calculates to

$$\begin{aligned} W(y_1,y_2) &= \mathrm{e}_{\lambda_1}(\cdot,t_0)\lambda_2\,\mathrm{e}_{\lambda_2}(\cdot,t_0) - \lambda_1\,\mathrm{e}_{\lambda_1}(\cdot,t_0)\,\mathrm{e}_{\lambda_2}(\cdot,t_0) \\ &= (\lambda_2 - \lambda_1)\,\mathrm{e}_{\lambda_1\oplus\lambda_2}(\cdot,t_0). \end{aligned}$$

Hence W is never zero if $\lambda_1 \neq \lambda_2$, i.e., if $\alpha^2 \neq 4\beta$. In this case we say that y_1 and y_2 form a *fundamental system* of (5.3), and the solution of any initial value problem

$$y^{\Delta\Delta} + \alpha y^\Delta + \beta y = 0, \quad y(t_0) = y_0, \quad y^\Delta(t_0) = y_0^\Delta \tag{5.4}$$

can be given as a linear combination of y_1 and y_2. Now we can state the main result on equations of the form (5.3), see [8].

Theorem 5.2. (i) *If* $\alpha^2-4\beta>0$, *put* $p=-\alpha/2$ *and* $q=\sqrt{\alpha^2-4\beta}/2$. *If* p *and* $\mu\beta-\alpha$ *are regressive, then a fundamental system of* (5.3) *is given by* $\cosh_{q/(1+\mu p)}(\cdot,t_0)\,\mathrm{e}_p(\cdot,t_0), \sinh_{q/(1+\mu p)}(\cdot,t_0)\mathrm{e}_p(\cdot,t_0)$, *with Wronskian* $q\mathrm{e}_{\mu\beta-\alpha}(\cdot,t_0)$, *and the solution of* (5.4) *is*

$$\left[y_0\cosh_{q/(1+\mu p)}(\cdot,t_0)+\frac{y_0^{\Delta}-py_0}{q}\sinh_{q/(1+\mu p)}(\cdot,t_0)\right]\mathrm{e}_p(\cdot,t_0).$$

(ii) *If* $\alpha^2-4\beta<0$, *put* $p=-\alpha/2$ *and* $q=\sqrt{4\beta-\alpha^2}/2$. *If* $p,q\in\mathbb{R}$ *and* p *is regressive, then a fundamental system of* (5.3) *is given by* $\cos_{q/(1+\mu p)}(\cdot,t_0)\,\mathrm{e}_p(\cdot,t_0)$, $\sin_{q/(1+\mu p)}(\cdot,t_0)\mathrm{e}_p(\cdot,t_0)$, *with Wronskian* $q\mathrm{e}_{\mu\beta-\alpha}(\cdot,t_0)$, *and the solution of* (5.4) *is*

$$\left[y_0\cos_{q/(1+\mu p)}(\cdot,t_0)+\frac{y_0^{\Delta}-py_0}{q}\sin_{q/(1+\mu p)}(\cdot,t_0)\right]\mathrm{e}_p(\cdot,t_0).$$

(iii) *If* $\alpha^2-4\beta=0$, *put* $p=-\alpha/2$. *If* p *is regressive, then a fundamental system of* (5.3) *is given by* $\mathrm{e}_p(t,t_0)$ *and* $\mathrm{e}_p(t,t_0)\int_{t_0}^t\frac{\Delta\tau}{1+p\mu(\tau)}$, *with Wronskian* $\mathrm{e}_{\mu p^2}(\cdot,t_0)$, *and the solution of* (5.4) *is*

$$\left[y_0+(y_0^{\Delta}-py_0)\int_{t_0}^t\frac{\Delta\tau}{1+p\mu(\tau)}\right]\mathrm{e}_p(t,t_0).$$

Another result concerns the *Euler–Cauchy equation*

$$t\sigma(t)y^{\Delta\Delta}+\alpha ty^{\Delta}+\beta y=0,\quad t>0, \tag{5.5}$$

where α and β are constant, and where we assume that the regressivity condition

$$1-\frac{\alpha\mu(t)}{\sigma(t)}+\frac{\beta\mu^2(t)}{t\sigma(t)}\neq 0 \tag{5.6}$$

holds. The associated *characteristic equation* is

$$\lambda^2+(\alpha-1)\lambda+\beta=0. \tag{5.7}$$

Theorem 5.3. *Suppose* (5.6). *If* (5.7) *has two distinct roots* λ_1 *and* λ_2, *then*

$$\mathrm{e}_{\lambda_1/t}(t,t_0)\quad\textit{and}\quad\mathrm{e}_{\lambda_2/t}(t,t_0)$$

is a fundamental system of (5.5). *If* $(\alpha-1)^2=4b^2$, *then we put* $p=(\alpha-1)/2$, *and*

$$\mathrm{e}_{p/t}(t,t_0)\quad\textit{and}\quad\mathrm{e}_{p/t}(t,t_0)\int_{t_0}^t\frac{\Delta\tau}{\tau+p\mu(\tau)}$$

is a fundamental system of (5.5).

For the more general problem

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t), \quad y(t_0) = y_0, \quad y^{\Delta}(t_0) = y_0^{\Delta} \tag{5.8}$$

we have the following variation of constants result:

Theorem 5.4. *If f is rd-continuous and if y_1 and y_2 form a fundamental system for the equation $y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = 0$, then the solution of* (5.8) *is given by*

$$c_1 y_1(t) + c_2 y_2(t) + \int_{t_0}^{t} \frac{y_2(t)y_1(\sigma(\tau)) - y_1(t)y_2(\sigma(\tau))}{W(y_1, y_2)(\sigma(\tau))} f(\tau)\Delta\tau,$$

where

$$c_1 = \frac{y_2^{\Delta}(t_0)y_0 - y_2(t_0)y_0^{\Delta}}{W(y_1, y_2)(t_0)} \quad \textit{and} \quad c_2 = \frac{y_1(t_0)y_0^{\Delta} - y_1^{\Delta}(t_0)y_0}{W(y_1, y_2)(t_0)}.$$

Let us now consider the Wronskian of any two solutions x_1 and x_2 of

$$x^{\Delta\Delta} + p(t)x^{\Delta^\sigma} + q(t)x^{\sigma} = 0. \tag{5.9}$$

We have

$$\begin{aligned}
W(x_1,x_2)^{\Delta} &= \det\begin{pmatrix} x_1 & x_2 \\ x_1^{\Delta} & x_2^{\Delta} \end{pmatrix}^{\Delta} = (x_1x_2^{\Delta} - x_1^{\Delta}x_2)^{\Delta} \\
&= x_1^{\sigma}x_2^{\Delta\Delta} + x_1^{\Delta}x_2^{\Delta} - x_1^{\Delta\Delta}x_2^{\sigma} - x_1^{\Delta}x_2^{\Delta} \\
&= x_1^{\sigma}x_2^{\Delta\Delta} - x_2^{\sigma}x_1^{\Delta\Delta} = \det\begin{pmatrix} x_1^{\sigma} & x_2^{\sigma} \\ x_1^{\Delta\Delta} & x_2^{\Delta\Delta} \end{pmatrix} \\
&= \det\begin{pmatrix} x_1^{\sigma} & x_2^{\sigma} \\ -px_1^{\Delta^\sigma} - qx_1^{\sigma} & -px_2^{\Delta^\sigma} - qx_2^{\sigma} \end{pmatrix} \\
&= \det\begin{pmatrix} x_1^{\sigma} & x_2^{\sigma} \\ -px_1^{\Delta^\sigma} & -px_2^{\Delta^\sigma} \end{pmatrix} = -\,p\det\begin{pmatrix} x_1^{\sigma} & x_2^{\sigma} \\ x_1^{\Delta^\sigma} & x_2^{\Delta^\sigma} \end{pmatrix} \\
&= -p\det\begin{pmatrix} x_1 & x_2 \\ x_1^{\Delta\Delta} & x_2^{\Delta\Delta} \end{pmatrix} = -\,pW(x_1,x_2)^{\sigma}.
\end{aligned}$$

Hence, by Theorem 5.1, we obtain the following *Abel's formula*:

Theorem 5.5. *If p is regressive and rd-continuous and if x_1 and x_2 solve* (5.9), *then*

$$W(x_1,x_2)(t) = W(x_1,x_2)(t_0)e_{\ominus p}(t,t_0).$$

A similar result holds for higher order equations of the form

$$x^{\Delta^n} + \sum_{k=1}^{n} q_k(t)\left(x^{\Delta^{n-k}}\right)^{\sigma} = 0. \tag{5.10}$$

It can be shown that initial value problems involving equation (5.10) have unique solutions provided the q_k are rd-continuous for $1 \leqslant k \leqslant n$ and q_1 is regressive, see [6]. Wronskians of n functions $x_1, \ldots, x_n$, all of them $n-1$ times differentiable, are defined in the natural way

$$W(x_1, \ldots, x_n) = \det \begin{pmatrix} x_1 & \cdots & x_n \\ x_1^{\Delta} & \cdots & x_n^{\Delta} \\ \vdots & & \vdots \\ x_1^{\Delta^{n-1}} & \cdots & x_n^{\Delta^{n-1}} \end{pmatrix}. \tag{5.11}$$

Theorem 5.6. *If q_1 is regressive and rd-continuous and if $x_1, \ldots, x_n$ solve* (5.10), *then*

$$W(x_1, \ldots, x_n)(t) = W(x_1, \ldots, x_n)(t_0) e_{\ominus q_1}(t, t_0).$$

The variations of constant result for higher order equations is stated in terms of

$$y^{\Delta^n} + \sum_{k=1}^{n} p_k(t) y^{\Delta^{n-k}} = f(t) \tag{5.12}$$

and reads as follows:

Theorem 5.7. *If $f, p_1, \ldots, p_n$ are rd-continuous and $-\sum_{k=1}^{n}(-\mu)^{k-1} p_k$ is regressive, and if $y_1, \ldots, y_n$ is a fundamental system for $y^{\Delta^n} + \sum_{k=1}^{n} p_k(t) y^{\Delta^{n-k}} = 0$, then all solutions of* (5.12) *are given by*

$$\sum_{k=1}^{n} c_k y_k(t) + \int_{t_0}^{t} \frac{W(\sigma(\tau), t)}{W(\sigma(\tau))} f(\tau) \Delta\tau,$$

where $c_1, \ldots, c_n$ are constants, $W = W(y_1, \ldots, y_n)$, and $W(\tau, t)$ is the determinant of the matrix in (5.11) *where the last row is replaced by $(y_1(t) \cdots y_n(t))$.*

For the remainder of this section we now consider *self-adjoint* equations of second order

$$(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0 \quad \text{with } p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}. \tag{5.13}$$

Clearly, by Theorem 5.5, if $p(t) \equiv 1$, then the Wronskian of any two solutions of (5.13) is constant. In case of (5.13), this is true for the *generalized Wronskian*

$$W(x_1, x_2) = \det \begin{pmatrix} x_1 & x_2 \\ p x_1^{\Delta} & p x_2^{\Delta} \end{pmatrix}$$

since

$$W(x_1, x_2)^{\Delta} = (x_1 p x_2^{\Delta} - x_2 p x_1^{\Delta})^{\Delta} = x_1^{\sigma}(p x_2^{\Delta})^{\Delta} - x_2^{\sigma}(p x_1^{\Delta})^{\Delta} = -x_1^{\sigma} q x_2^{\sigma} + x_2^{\sigma} q x_1^{\sigma} = 0.$$

Theorem 5.8. *The Wronskian of any two solutions of* (5.13) *is constant.*

Now assume that x solves (5.13) such that $x(t) \neq 0$ for all $t \in \mathbb{T}$. We make the *Riccati* substitution

$$r = \frac{px^\Delta}{x}$$

and find that

$$\begin{aligned} r^\Delta + q + \frac{r^2}{p + \mu r} &= \frac{(px^\Delta)^\Delta x - x^\Delta px^\Delta}{xx^\sigma} + q + \frac{p^2(x^\Delta)^2}{x^2} \Big/ \left(p + \frac{p\mu x^\Delta}{x} \right) \\ &= \frac{-qx^\sigma x - p(x^\Delta)^2}{xx^\sigma} + q + \frac{p^2(x^\Delta)^2}{x^2} \frac{x}{px^\sigma} = 0. \end{aligned}$$

Hence r solves the *Riccati equation*

$$r^\Delta + q(t) + \frac{r^2}{p(t) + \mu(t) r} = 0. \tag{5.14}$$

The following *Riccati equivalence* is shown in [1]:

Theorem 5.9. *There exists a solution x of* (5.13) *with $x(t) \neq 0$ for all $t \in \mathbb{T}$ iff* (5.14) *has a solution r (related by $r = px^\Delta/x$).*

Now suppose again that x solves (5.13) and is never zero. Let η be any differentiable function on $\mathbb{T}$. With $r = px^\Delta/x$ we have

$$\begin{aligned} (\eta^2 r)^\Delta + (p\eta^\Delta - r\eta)^2 \frac{x}{px^\sigma} &= \left(\frac{\eta^2}{x}\right)^\sigma (px^\Delta)^\Delta + \left(\frac{\eta^2}{x}\right)^\Delta px^\Delta + p\left(\eta^\Delta - \frac{x^\Delta \eta}{x}\right)^2 \frac{x}{x^\sigma} \\ &= -\left(\frac{\eta^2}{x}\right)^\sigma qx^\sigma + \left[\eta^\Delta \frac{\eta}{x} + \eta^\sigma \left(\frac{\eta}{x}\right)^\Delta\right] px^\Delta + p\left[\left(\frac{\eta}{x}\right)^\Delta\right]^2 xx^\sigma \\ &= -(\eta^\sigma)^2 q + \frac{\eta\eta^\Delta px^\Delta}{x} + p\left(\frac{\eta}{x}\right)^\Delta \left[\eta^\sigma x^\Delta + \left(\frac{\eta}{x}\right)^\Delta xx^\sigma\right] \\ &= -(\eta^\sigma)^2 q + \frac{\eta\eta^\Delta px^\Delta}{x} + p\left(\frac{\eta}{x}\right)^\Delta [\eta^\sigma x^\Delta + \eta^\Delta x - \eta x^\Delta] \\ &= -(\eta^\sigma)^2 q + \frac{\eta\eta^\Delta px^\Delta}{x} + p\left(\frac{\eta}{x}\right)^\Delta \eta^\Delta x^\sigma \\ &= p(\eta^\Delta)^2 - q(\eta^\sigma)^2. \end{aligned}$$

This so-called *Picone identity* can be used to prove the following *Jacobi condition* which is a result on *positive definiteness* of the quadratic functional

$$\mathscr{F}(\eta) = \int_a^b \{p(\eta^\Delta)^2 - q(\eta^\sigma)^2\}(t)\Delta t \quad \text{where } a, b \in \mathbb{T}.$$

Here, $\mathscr{F}$ is called positive definite if $\mathscr{F}(\eta) > 0$ for all nontrivial differentiable functions η with $\eta(a) = \eta(b) = 0$. We also call (5.13) *disconjugate* (on $[a,b]$) if the solution x of

$$(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0, \quad x(a) = 0, \quad x^{\Delta}(a) = \frac{1}{p(a)}$$

satisfies

$$px^{\sigma}x > 0 \quad \text{on } (a,b]^{\kappa}.$$

Theorem 5.10. *$\mathscr{F}$ is positive definite iff* (5.13) *is disconjugate.*

Finally, we present a Sturm separation theorem and a Sturm comparison theorem. Sturmian theory can be developed as follows: We say that a solution $\tilde{x}$ of (5.13) has *no focal points* in $(a,b]$ if

$$p\tilde{x}^{\sigma}\tilde{x} > 0 \quad \text{on } (a,b]^{\kappa} \quad \text{and} \quad p(a)\tilde{x}(a)\tilde{x}^{\sigma}(a) \geqslant 0.$$

Theorem 5.11. *If there exists a solution of* (5.13) *without focal points, then* (5.13) *is disconjugate.*

Consider another self-adjoint equation of the form (5.13),

$$(\tilde{p}(t)x^{\Delta})^{\Delta} + \tilde{q}(t)x^{\sigma} = 0. \tag{5.15}$$

Theorem 5.12. *Suppose $\tilde{p}(t) \leqslant p(t)$ and $\tilde{q}(t) \geqslant q(t)$ for all $t \in \mathbb{T}$. If* (5.14) *is disconjugate, then so is* (5.15).

6. Symplectic systems

Let us now rewrite (5.13) as a system by introducing $u = px^{\Delta}$. We find

$$x^{\Delta} = \frac{1}{p}u \quad \text{and} \quad u^{\Delta} = -qx^{\sigma} = -qx - \frac{\mu q}{p}u,$$

so with $z = \binom{x}{u}$ we have

$$z^{\Delta} = S(t)z \quad \text{where } S = \begin{pmatrix} 0 & \frac{1}{p} \\ -q & -\frac{\mu q}{p} \end{pmatrix}.$$

The matrix $I + \mu S$ has determinant

$$\det \begin{pmatrix} 1 & \frac{\mu}{p} \\ -\mu q & 1 - \frac{\mu^2 q}{p} \end{pmatrix} = 1 - \frac{\mu^2 q}{p} + \frac{\mu^2 q}{p} = 1$$

and hence S is regressive. Subject to the assumptions that p and q are rd-continuous, Theorem 3.6 yields the unique solvability of initial value problems involving $z^{\Delta} = S(t)z$. The above

2×2-matrix-valued function S has, besides being rd-continuous and regressive, another remarkable property, namely, we have with $\mathscr{T}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$S^T\mathscr{T}+\mathscr{T}S+\mu S^T\mathscr{T}S=\begin{pmatrix} q & 0 \\ \frac{\mu q}{p} & \frac{1}{p} \end{pmatrix}+\begin{pmatrix} -q & -\frac{\mu q}{p} \\ 0 & -\frac{1}{p} \end{pmatrix}+\mu\begin{pmatrix} q & 0 \\ \frac{\mu q}{p} & \frac{1}{p} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{p} \\ -q & -\frac{\mu q}{p} \end{pmatrix}$$

$$=\begin{pmatrix} 0 & -\frac{\mu q}{p} \\ \frac{\mu q}{p} & 0 \end{pmatrix}+\mu\begin{pmatrix} 0 & \frac{q}{p} \\ -\frac{q}{p} & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e.,

$$S^T\mathscr{T}+\mathscr{T}S+\mu S^T\mathscr{T}S=0. \tag{6.1}$$

In general, we call a $2n\times 2n$-matrix-valued function S *symplectic* (with respect to $\mathbb{T}$) if (6.1) holds with

$$\mathscr{T}=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The corresponding system

$$z^\Delta=S(t)z \quad \text{where } S \text{ satisfies (6.1) and is rd-continuous} \tag{6.2}$$

is called a *symplectic system*. If S is symplectic (w.r.t. $\mathbb{T}$), then it is regressive:

$$(I+\mu S)^T\mathscr{T}(I+\mu S)=\mathscr{T}+\mu(S^T\mathscr{T}+\mathscr{T}S+\mu S^T\mathscr{T}S)=\mathscr{T}. \tag{6.3}$$

Hence, by Theorem 3.6, initial value problems involving symplectic systems (6.2) have unique solutions. Now let z_1 and z_2 be any two solutions of (6.2). Then their (generalized) Wronskian is defined as

$$W(z_1,z_2)=z_1^T\mathscr{T}z_2.$$

Because of

$$\begin{aligned}W(z_1,z_2)^\Delta &=(z_1^\Delta)^T\mathscr{T}z_2^\sigma+z_1^T\mathscr{T}z_2^\Delta\\ &=(z_1^\Delta)^T\mathscr{T}(z_2+\mu z_2^\Delta)+z_1^T\mathscr{T}z_2^\Delta\\ &=z_1^TS^T\mathscr{T}(z_2+\mu Sz_2)+z_1^T\mathscr{T}Sz_2\\ &=z_1^T[S^T\mathscr{T}+\mathscr{T}S+\mu S^T\mathscr{T}S]z_2\\ &=0,\end{aligned}$$

we have, see [10]:

Theorem 6.1. *The Wronskian of any two solutions of* (6.2) *is constant.*

A *conjoined solution* of (6.2) is a $2n\times n$-matrix-valued solution Z of (6.2) with $Z^T\mathscr{T}Z\equiv 0$. The conjoined solution of (6.2) satisfying $Z(a)=\binom{0}{I}$ is called the *principal solution* of (6.2) (at a).

Example 6.1. First, S is symplectic with respect of $\mathbb{R}$ iff

$$S^T\mathscr{T} + \mathscr{T}S = 0,$$

i.e., iff $\mathscr{T}S$ is symmetric. Then, necessarily, S must be of the form

$$S = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \quad \text{with symmetric } B, C$$

and is called *Hamiltonian*. Next, S is symplectic with respect to $\mathbb{Z}$ iff, see (6.3),

$$(I+S)^T\mathscr{T}(I+S) = \mathscr{T},$$

i.e., iff $I+S$ is *symplectic*. Here, as usual, a matrix $2n \times 2n$-matrix M is called symplectic, if $M^T\mathscr{T}M = \mathscr{T}$. Note also that (6.3) shows that $I + \mu S$ is symplectic whenever S is symplectic with respect to $\mathbb{T}$. We also remark that symplectic systems and Hamiltonian systems are the same for $\mathbb{T} = \mathbb{R}$ but that there are more discrete symplectic systems than Hamiltonian difference systems, see [5].

Now, let us assume that (6.2) has a conjoined solution $Z = \binom{X}{U}$ such that X is invertible. We consider the Riccati substitution

$$R = UX^{-1}.$$

The calculation

$$\begin{aligned} R^\Delta &= U^\Delta(X^\sigma)^{-1} - UX^{-1}X^\Delta(X^\sigma)^{-1} \\ &= (U^\Delta - RX^\Delta)(X^\sigma)^{-1} \\ &= (I\ \ R)\mathscr{T}Z^\Delta(X^\sigma)^{-1} \\ &= (I\ \ R)\mathscr{T}SZ(X^\sigma)^{-1} \\ &= (I\ \ R)\mathscr{T}S(I+\mu S)^{-1}Z^\sigma(X^\sigma)^{-1} \\ &= -(I\ \ R)S^T\mathscr{T}\begin{pmatrix} I \\ R^\sigma \end{pmatrix} \end{aligned}$$

shows that R is a (symmetric) solution of the matrix Riccati equation

$$R^\Delta + (I\ \ R)S^T\mathscr{T}\begin{pmatrix} I \\ R^\sigma \end{pmatrix} = 0.$$

The precise statement of the Riccati equivalence (compare Theorem 5.9) may be found in [47, Theorem 3]. In [14], the generalization of Theorem 5.10 is derived:

Theorem 6.2. *Subject to a normality condition*, $\mathscr{F}$ *is positive definite iff* (6.2) *is disconjugate.*

In Theorem 6.2, the quadratic functional

$$\mathscr{F}(z)=\int_a^b \{z^T(S^T\mathscr{M}S-\mathscr{M})z\}(t)\Delta t \quad \text{with } \mathscr{M}=\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

is called positive definite, if $\mathscr{F}(z)>0$ for all z with $\mathscr{M}z^\Delta=\mathscr{M}Sz$, $\mathscr{M}z\not\equiv 0$, and $\mathscr{M}z(a)=\mathscr{M}z(b)=0$. System (6.2) is called disconjugate if the principal solution of (6.2) (at a) has no focal points in $(a,b]$, and this is defined as

$$\begin{cases} X(t) \text{ is invertible if } t\in(a,b] \text{ is left-dense or right-dense} \\ \operatorname{Ker} X(\sigma(t))\subset \operatorname{Ker} X(t),\ X(t)X^\dagger(\sigma(t))B(t)\geqslant 0 \quad \text{for all } t\in(a,b]^\kappa. \end{cases}$$

Here, Ker stands for the kernel and the dagger denotes the Moore–Penrose inverse of the matrix indicated, "$\geqslant 0$" means positive semidefinite, and B is the $n\times n$-matrix in the right upper corner of S.

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