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Multiplier-accelerator Models on Time Scales

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ABSTRACT

In this work we derive a linear second-order dynamic equation which describes multiplier-accelerator models on time scales. After we provide the general form of the dynamic equation, which considers both taxes and foreign trade, i.e., imports and exports, we give four special cases of this general multiplier-accelerator model: (1) Samuelson's basic multiplier-accelerator model. (2) We extend this model with the assumption that taxes are raised by the government and that these taxes are immediately reinvested by the government. (3) We give Hicks' extension of the basic multiplier-accelerator model as an example and (4) extend this model by allowing foreign trade in the next step. For each of these models we present the dynamic equation in both expanded and self-adjoint form and give examples for particular time scales.

Keywords: Time scales, multiplier–accelerator, dynamic equation, self-adjoint, economics.

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1 Introduction

In 1939, Samuelson combined in (Samuelson, 1939) the multiplier model with the acceleration principle. The acceleration principle is a theory which states that small changes in the demand for consume goods can generate large changes in the demand for investment (capital) goods needed for their production. In (Samuelson, 1939), he derived a second-order difference equation which describes the model of the combination of those two principles. In his interaction model, he assumes that the national income Y is dependent on the following three expenditure streams: Induced investment I, autonomous investment G (government expenditure), and consumption C. Autonomous investment is the part of investment that is totally independent of the current state of the economy, whereas the induced investment does depend on the present state of the economy. The consumption is supposed to be strictly proportional to the national income with a one period lag, i.e., (1) $C_t = bY_{t-1}$, where b is the propensity to spend money. If now one of the components G or I, i.e., autonomous or induced investment, is increased, then the *multiplier* doctrine states that the national income will also increase, but this increase is higher than the initial increase of investment. In (Samuelson, 1970), the author defined: "The multiplier is the number by which the change in investment must be multiplied in order to present us with the resulting change in income." The model also assumes that the induced investment is in a constant ratio with the increase of the consumption from the previous period to the current period, i.e., (2) $I_t = \beta\Delta C_{t-1} = \beta(C_t - C_{t-1})$. In this formula β denotes the *accelerator coefficient*. This is how the accelerator principle comes into the model. The *acceleration effect* is the impact of change in consumption on the investment. We also see from (2) that consumption has to continue increasing to make the investment stand still. Finally the equilibrium condition (3) $Y_t = C_t + I_t + G_t$ closes the model. Using (1) and (2), Samuelson derived from (3) the linear second-order difference equation

$$Y_t - b(1+\beta)Y_{t-1} + b\beta Y_{t-2} = G. \tag{1.1}$$

Of course this model can be made more complex for instance by assuming that the government raises taxes. In this work we assume that the government reinvests the taxes completely in the same period plus a constant rate $\bar{G}$. Later Hicks extended this model, by making further assumptions. This model differs from Samuelson's basic multiplier-accelerator model according to (Gandolfo, 1980, Chapter 6), in the following three main points:

1. The autonomous investment G is supposed to be of the form $G_t = A_0(1+g)^t$, where A_0 is the initial value of the autonomous investment and g the growth rate of the autonomous investment.

2. The accelerator-induced investment I does now not depend anymore only on the change in the consumption demand, but on the change of the total demand, i.e., on the change of the national income.

3. The induced investment is directly proportional to the increase of the national income from two periods before to the previous period, i.e., $\Delta Y_{t-2} = Y_{t-1} - Y_{t-2}$, and not $\Delta Y_{t-1} = Y_t - Y_{t-1}$.

Another way to make the models more realistic is to assume that we are in an open economy. With open economy we always mean that trade between countries takes place, i.e., we have two new streams, imports and exports. If we say closed economy, we always mean that the country does neither import goods from other countries nor does it export goods to other countries. The Hicksian model can be extended in a way that allows foreign trade, i.e., we suppose that the nation imports and exports goods. We assume that the imports of the current period M_t are directly proportional to the national income with a one period lag, i.e., $M_t = mY_{t-1}$. Moreover we presume that exports X_t grow with initial value X_0 and a constant growth rate x, i.e., $X_t = X_0(1+x)^t$. A summary of all important variables with explanations is provided in

Table 1. Our goal is to derive a model which generalizes several multiplier-accelerator models (the four which are mentioned in the abstract and explained above) on time scales where the forward jump operator is delta-differentiable. This means that we want to unify and extend the real and discrete cases of these models. The setup of this paper is as follows. In Section 2, we establish a general multiplier-accelerator model on time scales. Next, in Section 3, we apply our result to four different models and also show how the classical versions of these models follow from our general results as special cases. Examples for specific time scales are offered. Finally, in Section 4, we provide a short conclusion to summarize what has been done in the paper.

Time scales theory was originally introduced by Stefan Hilger in 1988 in his PhD thesis (Hilger, 1988), supervised by Bernd Aulbach. In his work, he started to unify the theory of differential equations and difference equations. In former times mathematicians always thought that a dynamical process is either of a continuous or a discrete nature. But there are examples where a unifying approach makes more sense. Applications in the fast growing time scales field can be found for example in biological, physical-, life- and social sciences, and economics. Atici et al. already examined some economic issues on time scales in (Atici, Biles and Lebedinsky, 2006) and (Atici and Uysal, 2008). In this work we omit a detailed introduction to time scales and recommend the interested reader to consult (Bohner and Peterson, 2001) and (Bohner and Peterson, 2003).

Table 1: Explanation of variables

stream or variable	explanation
Y	national income
I	induced investment
G	autonomous investment
C	consumption
b	propensity to spend money
β	accelerator coefficient
M	imports
X	exports
m	import rate
X_0	initial value of exports
x	growth rate of exports
A_0	initial value of autonomous investment (in Hicks' model)
g	growth rate of autonomous investment
A	sum of autonomous investment G and exports X
τ	tax rate
γ	either β or β/b depending on the model

2 The General Multiplier-accelerator Model on Time Scales

Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

In this definition we set $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has maximum t). If $f : \mathbb{T} \to \mathbb{R}$ is a function, then the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma = f \circ \sigma$. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We will also need the set $\mathbb{T}^\kappa$ which is defined in the following way: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Else, $\mathbb{T}^\kappa = \mathbb{T}$. Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then $f^\Delta(t)$ is defined as the number (provided that it exists) such that for every $\varepsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \varepsilon|\sigma(t) - s| \qquad \text{for all} \qquad s \in U.$$

We call this number $f^\Delta(t)$ the delta-derivative of f at t. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left sided limits exist (finite) at left-dense points in $\mathbb{T}$. We will write the set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ as $\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* given that

$$1 + \mu(t)p(t) \neq 0 \qquad \text{for all} \qquad t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Let $p, q \in \mathcal{R}$. Define the “circle minus” subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) := \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \qquad \text{for all} \qquad t \in \mathbb{T}^\kappa.$$

The time scale exponential function $e_p(\cdot, t_0)$ is defined for $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$ as the unique solution of the initial value problem

$$y^\Delta = p(t)y, \qquad y(t_0) = 1 \qquad \text{on } \mathbb{T}.$$

In our calculations we will also need the two useful formulas

$$Z^\sigma = Z + \mu Z^\Delta, \tag{2.1}$$

and (see (Bohner and Tisdell, 2005, Lemma 1))

$$Z^{\sigma\Delta} = \sigma^\Delta Z^{\Delta\sigma}. \tag{2.2}$$

Throughout this whole work we require that the forward jump operator σ is delta-differentiable and that $Y : \mathbb{T} \to \mathbb{R}$, $I : \mathbb{T} \to \mathbb{R}$, $C : \mathbb{T} \to \mathbb{R}$, $A : \mathbb{T} \to \mathbb{R}$ and $M : \mathbb{T} \to \mathbb{R}$. Furthermore we require that I, C, A, and M are delta-differentiable on $\mathbb{T}^\kappa$.

2.1 Expanded form of the general multiplier-accelerator model

Let us consider an economy where the national income depends on consumption, induced and autonomous investment, and import and exports. Furthermore suppose the consumption is strictly proportional to the national income with a one-period lag and the induced investment is in a constant ratio with the increase of the consumption from the previous period to the current period. Moreover we assume that the autonomous investment consists of a constant component $\bar{G}$ and an exponentially growing component $A_0 e_g(\cdot, t_0)$ with growth rate g and initial autonomous investment A_0, that the exports grow exponentially with growth rate x and initial export X_0, that the imports of the current period M_t are directly proportional to the national income with a one-period lag, and finally that the tax rate is currently τ.

Definition 2.1. We define the general multiplier-accelerator model on time scales with the following four axioms:

$$Y = C + I + A - M + \tau Y, \tag{2.3}$$

where $A := G + X$, with $G = \bar{G} + A_0 e_g(\cdot, t_0)$ and $X = X_0 e_x(\cdot, t_0)$,

$$I^\sigma = \gamma C^\Delta, \tag{2.4}$$

$$C^\sigma = b(1-\tau)Y + (\mu - 1)Y^\Delta, \tag{2.5}$$

$$M^\sigma = mY. \tag{2.6}$$

Lemma 2.1. *In the general multiplier-accelerator model we have*

$$I^\Delta = (1-\tau)Y^\Delta - \frac{1}{\gamma} I^\sigma - A^\Delta + M^\Delta \tag{2.7}$$

and

$$Y^\Delta = I^\sigma + A^\sigma - (1 + m - b(1-\tau))Y + \tau Y^\sigma. \tag{2.8}$$

Proof. From equations (2.3) and (2.4) we have

$$I^\Delta \overset{(2.3)}{=} Y^\Delta - C^\Delta - A^\Delta + M^\Delta - \tau Y^\Delta \overset{(2.4)}{=} (1-\tau)Y^\Delta - \frac{1}{\gamma} I^\sigma - A^\Delta + M^\Delta,$$

and using additionally equations (2.5), (2.6) and (2.1) we can derive

$$\begin{aligned} & I^\sigma + A^\sigma - (1 + m - b(1-\tau))Y + \tau Y^\sigma - Y^\Delta \\ \overset{(2.5)}{=} \; & I^\sigma + A^\sigma - Y - mY + b(1-\tau)Y + \tau Y^\sigma - \big(b(1-\tau)Y - C^\sigma + \mu Y^\Delta\big) \\ \overset{(2.6)}{=} \; & I^\sigma + A^\sigma + C^\sigma - M^\sigma + \tau Y^\sigma - \big(Y + \mu Y^\Delta\big) \\ \overset{(2.1)}{=} \; & (C + I + A - M + \tau Y)^\sigma - Y^\sigma \overset{(2.3)}{=} 0, \end{aligned}$$

which concludes the proof. □

Theorem 2.2. *Suppose that* σ^Δ *exists and let*

$$c := \sigma^\Delta \left(\frac{1}{\gamma} - 1 - \sigma^\Delta \mu \frac{1}{\gamma} \tau \right) \in \mathcal{R} \qquad \textit{and} \qquad d := b(1-\tau) - 1. \tag{2.9}$$

Then Y *satisfies*

$$Y^{\Delta\Delta} + \frac{c + \sigma^\Delta \frac{1}{\gamma} \mu (m - \tau - d) - d}{1 + \mu c} Y^\Delta + \frac{\sigma^\Delta \frac{1}{\gamma}(m - \tau - d)}{1 + \mu c} Y = \frac{\sigma^\Delta \frac{1}{\gamma}}{1 + \mu c} A^{\sigma\sigma}. \tag{2.10}$$

Proof. Using Lemma 2.1 and equations (2.2), (2.1) and (2.6), we can derive that

$$\begin{aligned}
Y^{\Delta\Delta} &\overset{(2.8)}{=} I^{\sigma\Delta} + A^{\sigma\Delta} - (1+m-b(1-\tau))Y^{\Delta} + \tau Y^{\sigma\Delta} \\
&\overset{(2.2)}{=} \sigma^{\Delta} I^{\Delta\sigma} + A^{\sigma\Delta} + (d-m)Y^{\Delta} + \sigma^{\Delta}\tau Y^{\Delta\sigma} \\
&\overset{(2.7)}{=} \sigma^{\Delta}\left[(1-\tau)Y^{\Delta\sigma} - \frac{1}{\gamma} I^{\sigma\sigma} - A^{\Delta\sigma} + M^{\Delta\sigma}\right] + A^{\sigma\Delta} \\
&\quad + (d-m)Y^{\Delta} + \sigma^{\Delta}\tau Y^{\Delta\sigma} \\
&\overset{(2.8)}{=} \sigma^{\Delta}(1-\tau)Y^{\Delta\sigma} - \sigma^{\Delta}\frac{1}{\gamma}\left(Y^{\Delta\sigma} - A^{\sigma\sigma} + (1+m-b(1-\tau))Y^{\sigma} - \tau Y^{\sigma\sigma}\right) \\
&\quad - \sigma^{\Delta}A^{\Delta\sigma} + \sigma^{\Delta}M^{\Delta\sigma} + A^{\sigma\Delta} + (d-m)Y^{\Delta} + \sigma^{\Delta}\tau Y^{\Delta\sigma} \\
&\underset{(2.2)}{\overset{(2.1)}{=}} \sigma^{\Delta}\left(1-\frac{1}{\gamma}\right)(Y^{\Delta} + \mu Y^{\Delta\Delta}) + \sigma^{\Delta}\frac{1}{\gamma}A^{\sigma\sigma} + \sigma^{\Delta}\frac{1}{\gamma}(d-m)(Y + \mu Y^{\Delta}) \\
&\quad + \sigma^{\Delta}\frac{1}{\gamma}\tau(Y + \mu Y^{\Delta} + \mu\sigma^{\Delta}(Y^{\Delta} + \mu Y^{\Delta\Delta})) + M^{\sigma\Delta} + (d-m)Y^{\Delta} \\
&\overset{(2.6)}{=} \sigma^{\Delta}\left(1-\frac{1}{\gamma}\right)(Y^{\Delta} + \mu Y^{\Delta\Delta}) + \sigma^{\Delta}\frac{1}{\gamma}A^{\sigma\sigma} + \sigma^{\Delta}\frac{1}{\gamma}(d-m)(Y + \mu Y^{\Delta}) \\
&\quad + \sigma^{\Delta}\frac{1}{\gamma}\tau(Y + \mu Y^{\Delta} + \mu\sigma^{\Delta}(Y^{\Delta} + \mu Y^{\Delta\Delta})) + dY^{\Delta} \\
&= -\mu c Y^{\Delta\Delta} + \left[-c + \sigma^{\Delta}\frac{1}{\gamma}\mu(d-m+\tau) + d\right]Y^{\Delta} + \sigma^{\Delta}\frac{1}{\gamma}(d-m+\tau)Y \\
&\quad + \sigma^{\Delta}\frac{1}{\gamma}A^{\sigma\sigma},
\end{aligned}$$

which completes the proof. □

Remark 2.1. If we assume $Y = \bar{Y}$ to be constant, we obtain a particular solution of (2.10)

$$\bar{Y} = \frac{A^{\sigma\sigma}}{m-\tau-d}, \tag{2.11}$$

which is the equilibrium value of the national income. The deviations from this equilibrium value will be given by the general solution of the corresponding homogeneous equation. The factor $1/[m-\tau-d]$ in (2.11) is called the *multiplier coefficient*. We know that $\sigma(t) = \mu(t) + t$, i.e., if μ is constant, then $\sigma^{\Delta}(t) = \mu^{\Delta}(t) + 1 = 1$. We can solve the homogeneous part of (2.10) by finding the roots of the corresponding characteristic equation, provided that the homogeneous equation is regressive. Once we find the two solutions λ_1 and λ_2 of the corresponding characteristic equation of (2.10), we know that the general solution of (2.10) is of the form

$$Y = a_1 e_{\lambda_1}(\cdot, t_0) + a_2 e_{\lambda_2}(\cdot, t_0) + \frac{A^{\sigma\sigma}}{m-\tau-d}. \tag{2.12}$$

2.2 Self-adjoint form of the general multiplier-accelerator model

Since the expanded form of the second-order linear dynamic equation (2.10) seems rather complicated, we derive now the self-adjoint form of (2.10). The next theorem tells us under which conditions a dynamic equation in expanded form can be transformed into an equation of the form

$$(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = 0, \tag{2.13}$$

and more importantly it tells us how to transform it into form (2.13).

Theorem 2.3. *((Bohner and Peterson, 2001, Theorem 4.12)) If $a_1, a_2 \in \mathrm{C_{rd}}$ and $\mu a_2 - a_1 \in \mathcal{R}$, then we can write the second-order dynamic equation*

$$Y^{\Delta\Delta} + a_1(t)Y^{\Delta} + a_2(t)Y = 0$$

in self-adjoint form (2.13)*, where (with $t_0 \in \mathbb{T}^\kappa$)*

$$p = e_\alpha(\cdot, t_0), \qquad \alpha = \ominus(\mu a_2 - a_1), \qquad \textit{and} \qquad q = (1 + \mu\alpha)\, p a_2.$$

Theorem 2.4. *Let c and d be as in* (2.9)*. Suppose $\sigma^\Delta \in \mathrm{C_{rd}}$ and $c,\, d \in \mathcal{R}$. Then the corresponding homogeneous equation of* (2.10) *has the self-adjoint form*

$$\left(e_\alpha(\cdot, t_0)Y^\Delta\right)^\Delta(t) + \sigma^\Delta(t)\frac{1}{\gamma}[(m-\tau)\ominus d](t)e_\alpha(t, t_0)Y^\sigma(t) = 0, \tag{2.14}$$

where

$$\alpha = c \ominus d.$$

Proof. By Theorem 2.3 we have

$$\begin{aligned} \alpha &= \ominus\left(\frac{\sigma^\Delta \frac{1}{\gamma}\mu(m-\tau-d)}{1+\mu c} - \frac{c + \sigma^\Delta\frac{1}{\gamma}\mu(m-\tau-d) - d}{1+\mu c}\right) \\ &= \ominus\left(\frac{d-c}{1+\mu c}\right) = \ominus(d \ominus c) = c \ominus d, \end{aligned}$$

where we have used (Bohner and Peterson, 2001, Exercise 2.28). Thus we get (using (Bohner and Peterson, 2001, Theorem 2.36))

$$p = e_\alpha(\cdot, t_0) = e_{c\ominus d}(\cdot, t_0) = \frac{e_c(\cdot, t_0)}{e_d(\cdot, t_0)}.$$

Furthermore

$$\begin{aligned} q &= (1+\mu\alpha)\, p a_2 = [1 + \mu(c\ominus d)]pa_2 = \frac{1+\mu c}{1+\mu d}pa_2 = p\sigma^\Delta\frac{1}{\gamma}\frac{m-\tau-d}{1+\mu d} \\ &= p\sigma^\Delta\frac{1}{\gamma}[(m-\tau)\ominus d]. \end{aligned}$$

This completes the proof. ☐

3 Applications

3.1 The tax-extended version of the basic multiplier-accelerator model

In this section we set $m = 0$, $\gamma = \beta$, $A_0 = 0$ and $X_0 = 0$.

Theorem 3.1 (Expanded Form). *Suppose that σ^Δ exists and let*

$$c := \sigma^\Delta\left(\frac{1}{\beta} - 1 - \sigma^\Delta\mu\frac{1}{\beta}\tau\right) \in \mathcal{R} \qquad \textit{and} \qquad d := b(1-\tau) - 1. \tag{3.1}$$

Then Y satisfies

$$Y^{\Delta\Delta} + \frac{c - \sigma^\Delta\frac{1}{\beta}\mu(\tau+d) - d}{1+\mu c}Y^\Delta - \frac{\sigma^\Delta\frac{1}{\beta}(\tau+d)}{1+\mu c}Y = \frac{\sigma^\Delta\frac{1}{\beta}}{1+\mu c}\bar{G}. \tag{3.2}$$

Proof. The proof follows directly from Theorem 2.2. ☐

Theorem 3.2 (Self-Adjoint Form). *Let c and d be as in* (3.1). *Suppose $\sigma^\Delta \in \mathrm{C_{rd}}$ and $c,\, d \in \mathcal{R}$. Then the corresponding homogeneous equation of* (3.2) *has the self-adjoint form*

$$\left(e_\alpha(\cdot, t_0)Y^\Delta\right)^\Delta (t) + \sigma^\Delta(t)\frac{1}{\beta}[(-\tau) \ominus (b(1-\tau)-1)](t)e_\alpha(t, t_0)Y^\sigma(t) = 0, \tag{3.3}$$

where

$$\alpha = \sigma^\Delta \left(\frac{1}{\beta}(1-\sigma^\Delta\mu\tau) - 1\right) \ominus (b(1-\tau)-1) = c \ominus d.$$

Proof. The proof follows immediately from Theorem 2.4. ☐

Example 3.3. (i) If $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, then equation (3.3) can be written as

$$\left(e^{(\frac{1}{\beta}-b(1-\tau))t}Y'\right)' + \frac{1}{\beta}(1-b)(1-\tau)e^{(\frac{1}{\beta}-b(1-\tau))t}Y = 0.$$

(ii) If $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$, we obtain the expanded form

$$\Delta\Delta Y_t + (2 - b(1+\beta))\Delta Y_t + (1-b)Y_t = \frac{\bar{G}}{1-\tau}$$

or using the usual substitutions we arrive at (see (Gandolfo, 1980, Exercise 6.1.b))

$$Y_{t+2} - b(1+\beta)Y_{t+1} + \beta b Y_t = \frac{\bar{G}}{1-\tau}.$$

3.2 Samuelson's basic multiplier-accelerator model

Now we set $m = 0$, $\tau = 0$, $A_0 = 0$, $X_0 = 0$ and $\gamma = \beta$. Looking at the general multiplier-accelerator model, we can derive the following theorems. Samuelson's multiplier-accelerator model can be described on time scales with the following linear second-order dynamic equation.

Theorem 3.4 (Expanded Form). *Suppose that σ^Δ exists and let*

$$c := \sigma^\Delta\left(\frac{1}{\beta} - 1\right) \in \mathcal{R} \qquad \textit{and} \qquad d := b - 1. \tag{3.4}$$

Then Y satisfies

$$Y^{\Delta\Delta} + \frac{c - d\left(1 + \sigma^\Delta \frac{1}{\beta}\mu\right)}{1+\mu c}Y^\Delta - \frac{\sigma^\Delta \frac{1}{\beta} d}{1+\mu c}Y = \frac{\sigma^\Delta \frac{1}{\beta}}{1+\mu c}G^{\sigma\sigma}. \tag{3.5}$$

Proof. The proof follows directly from Theorem 3.1 with $\tau = 0$. ☐

The next theorem tells us how equation (3.5) can be written in self-adjoint form.

Theorem 3.5 (Self-Adjoint Form). *Let c and d be as in* (3.4). *Suppose $\sigma^\Delta \in \mathrm{C_{rd}}$ and $c,\, d \in \mathcal{R}$. Then the corresponding homogeneous equation of* (3.5) *has the self-adjoint form*

$$\left(e_\alpha(\cdot, t_0)Y^\Delta\right)^\Delta (t) + \sigma^\Delta(t)\frac{1}{\beta}(\ominus(b-1))(t)e_\alpha(t, t_0)Y^\sigma(t) = 0, \tag{3.6}$$

where

$$\alpha = \sigma^\Delta\left(\frac{1}{\beta} - 1\right) \ominus (b-1).$$

Proof. The proof follows immediately from Theorem 3.2 with $\tau = 0$. □

Example 3.6. (i) In the discrete case ($\mathbb{T} = \mathbb{Z}$), we have $\mu = 1$ and $\sigma^\Delta = 1$. Thus equation (3.5) turns into

$$\Delta\Delta Y_t + (2 - b(1+\beta))\Delta Y_t + (1-b)Y_t = G_{t+2}, \tag{3.7}$$

If we substitute $\Delta Y_t = Y_{t+1} - Y_t$ and $\Delta\Delta Y_t = Y_{t+2} - 2Y_{t+1} + Y_t$ in (3.7), then equation (3.7) turns into

$$Y_{t+2} - b(1+\beta)Y_{t+1} + b\beta Y_t = G_{t+2},$$

i.e., the homogeneous part of the equation is the same as in the model without taxes. This is the linear second-order difference equation Samuelson derived in (Samuelson, 1970).

(ii) In the continuous case ($\mathbb{T} = \mathbb{R}$), we have $\mu = 0$ and $\sigma^\Delta = 1$. This means that equation (3.5) has now the form

$$Y'' + \left(\frac{1}{\beta} - b\right) Y' + \frac{1}{\beta}(1-b)Y = \frac{G}{\beta},$$

compare (Puu and Sushko, 2006, Chapters 3.8 and 3.9).

(iii) If $\mathbb{T} = q^{\mathbb{N}}$, then $\mu = (q-1)t$ and $\sigma^\Delta = q$, i.e., equation (3.5) can be rewritten as

$$Y^{\Delta\Delta} + \frac{q\frac{1}{\beta}(1-b)(q-1)t - q(1-\frac{1}{\beta}) + 1 - b}{1 - q(1-\frac{1}{\beta})(q-1)t} Y^\Delta + \frac{q\frac{1}{\beta}(1-b)}{1 - q(1-\frac{1}{\beta})(q-1)t} Y$$
$$= \frac{q\frac{1}{\beta}}{1 - q(1-\frac{1}{\beta})(q-1)t} G\left(q^2 t\right).$$

3.3 The Hicksian extension in open economies

In this section we set $\tau = 0$, $\gamma = \frac{\beta}{b}$ and $\bar{G} = 0$. The Hicksian extension assumes the autonomous investment to grow exponentially with initial value A_0 and constant growth rate g. Furthermore it is assumed that the induced investment I does not depend anymore only on the change in consumption demand, it now depends on the change of the total demand. For more details, the interested reader might consult (Gandolfo, 1980).

Theorem 3.7 (Expanded Form). *Suppose that σ^Δ exists and let*

$$c := \sigma^\Delta \left(\frac{1}{\gamma} - 1\right) \in \mathcal{R} \qquad \textit{and} \qquad d := b - 1. \tag{3.8}$$

Then Y satisfies

$$Y^{\Delta\Delta} + \frac{c + \sigma^\Delta \frac{1}{\gamma}\mu(m-d) - d}{1+\mu c} Y^\Delta + \frac{\sigma^\Delta \frac{1}{\gamma}(m-d)}{1+\mu c} Y = \frac{\sigma^\Delta \frac{1}{\gamma}}{1+\mu c} A^{\sigma\sigma}, \tag{3.9}$$

where $A^{\sigma\sigma} = A_0 e_g^{\sigma\sigma}(\cdot, t_0) + X_0 e_x^{\sigma\sigma}(\cdot, t_0)$.

Proof. The proof follows directly from Theorem 2.2. □

Theorem 3.8 (Self-Adjoint Form). *Let c and d be as in* (3.8)*. Suppose $\sigma^\Delta \in \mathrm{C_{rd}}$ and $c,\ d \in \mathcal{R}$. Then the corresponding homogeneous equation of* (3.9) *has the self-adjoint form*

$$\left(e_\alpha(\cdot, t_0)Y^\Delta\right)^\Delta(t) + \sigma^\Delta(t)\frac{1}{\gamma}[m \ominus (b-1)](t)e_\alpha(t, t_0)Y^\sigma(t) = 0, \tag{3.10}$$

where

$$\alpha = \sigma^\Delta\left(\frac{1}{\gamma} - 1\right) \ominus (b-1) = c \ominus d.$$

Proof. The proof follows immediately from Theorem 2.4. □

Example 3.9. (i) If $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$, then equation (3.9) has the following form.

$$\Delta\Delta Y_t + \frac{\frac{1}{\gamma}(2+m-b) - b}{\frac{1}{\gamma}}\Delta Y_t + (1-b+m)Y_t = A_0(1+g)^{t+2} + X_0(1+x)^{t+2},$$

or applying the usual substitutions we obtain (see (Gandolfo, 1980, Exercise 6.3))

$$Y_{t+2} + (m - (b+\beta)) + \beta Y_t = A_{t+2}.$$

(ii) If $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, then equation (3.9) turns into

$$Y'' + \left(\frac{1}{\gamma} - b\right)Y' + \frac{1}{\gamma}(1-b+m)Y = \frac{1}{\gamma}\left(A_0e^{gt} + X_0e^{xt}\right) = \frac{1}{\gamma}A(t).$$

3.4 The Hicksian extension in closed economies

In this section we set $m = 0$, $\tau = 0$, $\bar{G} = 0$, $X_0 = 0$ and $\gamma = \frac{\beta}{b}$.

Theorem 3.10 (Expanded Form). *Suppose that σ^Δ exists and let*

$$c := \sigma^\Delta\left(\frac{1}{\gamma} - 1\right) \in \mathcal{R} \qquad \text{and} \qquad d := b - 1. \tag{3.11}$$

Then Y satisfies

$$Y^{\Delta\Delta} + \frac{c - d\left(1 + \sigma^\Delta\frac{1}{\gamma}\mu\right)}{1+\mu c}Y^\Delta - \frac{\sigma^\Delta\frac{1}{\gamma}d}{1+\mu c}Y = \frac{\sigma^\Delta\frac{1}{\gamma}}{1+\mu c}A_0e_g^{\sigma\sigma}(\cdot, t_0). \tag{3.12}$$

Proof. The proof follows directly from Theorem 3.7 with $m = 0$ and $X_0 = 0$. □

Theorem 3.11 (Self-Adjoint Form). *Let c and d be as in* (3.11)*. Suppose $\sigma^\Delta \in \mathrm{C_{rd}}$ and $c,\ d \in \mathcal{R}$. Then the corresponding homogeneous equation of* (3.12) *has the self-adjoint form*

$$(e_\alpha(\cdot, t_0)\,Y^\Delta)^\Delta(t) + \sigma^\Delta(t)\frac{1}{\gamma}(\ominus(b-1))(t)e_\alpha(t, t_0)Y^\sigma(t) = 0, \tag{3.13}$$

where

$$\alpha = \sigma^\Delta\left(\frac{1}{\gamma} - 1\right) \ominus (b-1) = c \ominus d.$$

Proof. The proof follows directly from Theorem 3.8 with $m = 0$ and $X_0 = 0$. □

Example 3.12. (i) If $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$, then equation (3.12) turns into

$$\Delta\Delta Y_t + (2 - b(1+\gamma))\Delta Y_t + (1-b)Y_t = A_0(1+g)^{t+2},$$

or (see (Gandolfo, 1980, Exercise 6.2))

$$Y_{t+2} - (b+\beta)Y_{t+1} + \beta Y_t = A_0(1+g)^{t+2}.$$

(ii) If $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, then equation (3.12) can be written as

$$Y'' + \left(\frac{1}{\gamma} - b\right)Y' + \frac{1}{\gamma}(1-b)Y = \frac{A_0 e^{gt}}{\gamma}.$$

(iii) If $\mathbb{T} = q^{\mathbb{N}}$, then $\mu = (q-1)t$ and $\sigma^{\Delta} = q$, i.e., equation (3.12) can be rewritten as

$$Y^{\Delta\Delta} + \frac{q\frac{1}{\gamma}(1-b)(q-1)t - q(1-\frac{1}{\gamma}) + 1 - b}{1 - q(1-\frac{1}{\gamma})(q-1)t}Y^{\Delta} + \frac{q\frac{1}{\gamma}(1-b)}{1-q(1-\frac{1}{\gamma})(q-1)t}Y$$
$$= \frac{q\frac{1}{\gamma}}{1-q(1-\frac{1}{\gamma})(q-1)t}A_0 e_g^{\sigma\sigma}(t,t_0).$$

4 Summary

This paper has considered the problem of unification of continuous and discrete multiplier-accelerator models. To accomplish this goal, the relatively new theory of dynamic equations on time scales was employed. This theory not only unifies the continuous and the discrete cases, but also extends those to other cases "in between". We presented a general multiplier-accelerator model on a time scale in both the expanded and self-adjoint form and used it to derive four other multiplier-accelerator models on a time scale: Samuelson's basic multiplier-accelerator model, the tax-extended version of the basic multiplier-accelerator, the Hicksian extension in closed economies, and the Hicksian extension in open economies. For each of the four models, we gave various examples using different time scales, and we also showed how they reduce to the classical models when the time scale is chosen to be the set of all integers.

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