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ONE-DIMENSIONAL INVERSE LIMITS WITH SET-VALUED FUNCTIONS

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Abstract. In this paper we show that an inverse limit is 1-dimensional whenever the bonding functions are upper semi-continuous from $[0, 1]$ into $C([0, 1])$ and $\dim(G(f_i \circ f_{i+1} \circ \cdots \circ f_j)) = 1$ for all integers i and j, $1 \le i \le j$. We apply this result to show that such an inverse limit is treelike and also to show that unions of certain finite collections of interval-valued upper semi-continuous functions on $[0, 1]$ produce treelike continua. The former extends treelikeness of inverse limits with set-valued functions to a larger class of bonding functions, while the latter generalizes known results for unions of two mappings.

1. Introduction

Inverse limits with mappings on $[0, 1]$ produce continua having dimension not greater than 1. On the other hand, inverse limits on $[0, 1]$ with a single set-valued function can produce infinite dimensional continua as well as continua of any finite dimension. For example, using the function $f : [0, 1] \to C([0, 1])$ given by $f(0) = [0, 1]$ and $f(t) = 0$ for $t > 0$ as a single bonding function produces an infinite dimensional continuum [4, Example 2.3]; the function given by $f(t) = 0$ for $0 \le t < 1/2$, $f(1/2) = [0, 1/2]$, $f(t) = 1/2$ for $1/2 < t < 1$, and $f(1) = [1/2, 1]$ produces a two-dimensional continuum [4, Example 5.3]. In 2011, Van Nall published theorems addressing the dimension of inverse limits with set-valued functions on finite dimensional compact metric spaces [11]. For functions on $[0, 1]$, Nall's results show that (1) an inverse limit with a sequence of upper semi-continuous functions each having 0-dimensional

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values has dimension 0 or 1 and (2) an inverse limit with a sequence of upper semi-continuous functions each having 0-dimensional point-inverses has dimension 0 or 1. While investigating treelikeness of inverse limits on $[0,1]$, we considered interval-valued upper semi-continuous functions on $[0,1]$ that may have flat spots on their graphs [6]. Roughly speaking, we showed that for a sequence of upper semi-continuous interval-valued functions on $[0,1]$, if no flat spot iterates under compositions into the closure of the set of points where a bonding function has nondegenerate values, then the dimension of the inverse limit is 0 or 1. In [6, Example 5.3], we gave an example of a function having nondegenerate interval values and infinitely many flat spots that produces a 1-dimensional continuum. We have constructed an unpublished example of a function on $[0,1]$ to which Nall's point-inverse theorem applies, but [6, Theorem 4.2] does not (the graph of the function simply contains vertical lines attached to the graph of the identity at points corresponding to the dyadic rationals similar to the graph in Example 6.1 of section 6). This unsatisfying dichotomy of results about dimension led us to re-examine these issues. In Theorem 4.3 of this paper, we prove that the dimension of the inverse limit is 0 or 1 for any inverse limit with a sequence of upper semi-continuous functions from $[0,1]$ into $C([0,1])$ such that $\dim(G(f_{i\,j})) = 1$ for all integers i and j, $1 \leq i \leq j$. We apply this result to improve an earlier theorem on treelikeness of inverse limits of functions that are certain unions of two mappings on $[0,1]$ (see Theorem 5.6) as well as to show that sequences of interval-valued functions on $[0,1]$ have treelike inverse limits if the dimension of the graphs of the compositions $f_{i\,j}$ does not exceed 1 (see Theorem 5.2).

2. Definitions and Background

By a *compactum* we mean a compact metric space. If X is a compactum, we denote the collection of closed subsets of X by 2^X; $C(X)$ denotes the connected elements of 2^X. If each of X and Y is a compactum, a function $f : X \to 2^Y$, sometimes denoted $f : X \nearrow Y$, is said to be *upper semi-continuous at the point x of* X provided that if V is an open subset of Y that contains $f(x)$, then there is an open subset U of X containing x such that if t is a point of U, then $f(t) \subseteq V$. A function $f : X \to 2^Y$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of X. If $f : X \to 2^Y$ is a set-valued function, by the *graph* of f, denoted $G(f)$, we mean $\{(x, y) \in X \times Y \mid y \in f(x)\}$. It is known that if X and Y are compacta and M is a subset of $X \times Y$ such that X is the projection of M to its set of first coordinates, then M is closed if and only if M is the graph of an upper semi-continuous function [8, Theorem 2.1]. In the case that f is upper semi-continuous and single-valued, i.e., $f(t)$ is degenerate for each $t \in X$, f is a continuous function. We call a continuous function a *mapping.* If X and Y are compacta and $f : X \to 2^Y$ is a set-valued function, we say that f is *lower semi-continuous* at $x \in X$ provided it is true that if $x_1, x_2, x_3, \ldots$ is a sequence of points of X converging to the point x and y is a point of $f(x)$, then there is a sequence $y_1, y_2, y_3, \ldots$ of points of Y that converges to $y \in Y$ such that $y_i \in f(x_i)$ for each $i \in \mathbb{N}$. A point of continuity of a set-valued function is a point at which the function is both upper and lower semi-continuous. Thus, for an upper semi-continuous function, a point at which it is not continuous is a point at which it is not lower semi-continuous. A subset A of a topological space X is *nowhere dense in* X provided the closure of A does not contain an open set. A subset A of a topological space X is of the *first category* provided A is the union of countably many nowhere dense sets. The following theorem may be found in [10, p. 71, Corollary 1].

Theorem 2.1. *If X and Y are compacta and $f : X \to 2^Y$ is upper semi-continuous, then $\{x \in X \mid f$ is not lower semi-continuous at $x\}$ is of the first category.*

We denote by $\mathbb{N}$ the set of positive integers. If $\boldsymbol{s} = s_1, s_2, s_3, \ldots$ is a sequence, we normally denote the sequence in boldface type and its terms in italics. Suppose $\boldsymbol{X}$ is a sequence of compacta each having diameter bounded by 1 and $f_n : X_{n+1} \nearrow X_n$ is an upper semi-continuous function for each $n \in \mathbb{N}$. By the *inverse limit* of $\boldsymbol{f}$, denoted $\varprojlim \boldsymbol{f}$, we mean $\{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for each positive integer $i\}$; we call the pair $\{\boldsymbol{X}, \boldsymbol{f}\}$ an inverse sequence. For an inverse sequence $\{\boldsymbol{X}, \boldsymbol{f}\}$ the following notation is convenient: If i and j are positive integers with $i < j$, $f_{i\,j} = f_i \circ f_{i+1} \cdots f_{j-1}$; thus, $f_{i\,j} : X_j \nearrow X_i$. It is convenient to denote by $f_{i\,i}$ the identity on X_i. Inverse limits are nonempty and compact [8, Theorem 3.2]; they are metric spaces being subsets of the metric space $\prod_{i>0} X_i$. We use the metric d on this product given by $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i>0} d_i(x_i, y_i)/2^i$. Because every metric space has an equivalent metric that is bounded by 1, we assume throughout that the metrics on our spaces are bounded by 1. In the case that each f_n is a mapping, the definition of the inverse limit reduces to the usual definition of an inverse limit on compacta with mappings (it is for this reason that we use the term inverse limit in discussing the corresponding construction using set-valued bonding functions). If $A \subseteq \mathbb{N}$, we denote by p_A the projection of $\prod_{n>0} X_n$ onto $\prod_{n\in A} X_n$ given $p_A(\boldsymbol{x}) = \boldsymbol{y}$ provided $y_i = x_i$ for each $i \in A$. If $A = \{n\}$, p_A is normally denoted p_n. In the case that $A \subseteq B \subseteq \mathbb{N}$, we normally also denote the restriction of p_A to $\prod_{n\in B} X_n$ by p_A, inferring by context that we are using this restriction. We denote the projection from the inverse limit into the ith factor space by π_i and, more generally, for $A \subseteq \mathbb{N}$, we denote by π_A the restriction of p_A to the inverse limit.

A set traditionally used in the proof that $\varprojlim \boldsymbol{f}$ is nonempty and compact is $\{\boldsymbol{x} \in \prod_{k>0} X_k \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$. Because this set was originally denoted G_n, we adopt and use throughout this article the notation $G'_n = \{\boldsymbol{x} \in \prod_{k=1}^{n+1} X_k \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$. Note that for $A = \{1, 2, \ldots, n+1\}$, $G'_n = p_A(G_n)$. In a recent paper [7], we observed that inverse limits on compacta with upper semi-continuous bonding functions is homeomorphic to an inverse limit on the sequence $G'_1, G'_2, G'_3, \ldots$ with bonding functions that are mappings. One consequence of this is that, in order to show that an inverse limit with set-valued functions has dimension 1, it is sufficient to show that G'_n is 1-dimensional for each positive integer n.

3. Properties of Upper Semi-Continuous Functions

The following theorem is not difficult to prove. A proof may be found in [7, Theorem 2.3].

Theorem 3.1. *Suppose each of f and g is an upper semi-continuous function from $[0,1]$ into $C([0,1])$. Then $g \circ f$ is an upper semi-continuous function from $[0,1]$ into $C([0,1])$.*

Theorem 3.2. *Suppose $f : [0,1] \to C([0,1])$ is upper semi-continuous and $\dim((G(f))) = 1$. If x is a point of $[0,1]$ and $f(x)$ is nondegenerate, then f is not lower semi-continuous at x.*

Proof. Suppose x is a point of $[0,1]$ and $f(x)$ is the nondegenerate interval $[a,b]$. Let p denote the midpoint of $[a,b]$. Because $\dim((G(f))) = 1$, $G(f)$ does not contain an open set. Thus, if O is an open set containing (x,p), there is a point (t,z) that belongs to O and not to $G(f)$. Choose a sequence $\boldsymbol{U}$ of open sets closing down on (x,p) so that no term of $\boldsymbol{U}$ contains a point having second coordinate 0 or 1; for each i, U_i contains a point (t_i, z_i) not in $G(f)$. The sequence $t_1, t_2, t_3, \ldots$ converges to x. Because z_i separates $[0,1]$ and $f(t_i)$ is connected for each i, $f(t_i) \subseteq [0, z_i)$ or $f(t_i) \subseteq (z_i, 1]$. Assume for infinitely many i, $f(t_i) \subseteq (z_i, 1]$. There exists a subsequence $\boldsymbol{s}$ of $\boldsymbol{t}$ and a point c, $a < c < b$, such that $f(s_i) \subseteq (c, 1]$ for each i. Because $\boldsymbol{s}$ is a subsequence of $\boldsymbol{t}$, $\boldsymbol{s}$ converges to x. However, if $y_i \in f(s_i)$ for each i, $y_1, y_2, y_3, \ldots$ does not converge to a; i.e., f is not lower semi-continuous at x. □

Recalling from Theorem 2.1 that the set of points at which an upper semi-continuous function is not continuous is a set of the first category, we have the following corollary to Theorem 3.2.

Corollary 3.3. *If $f : [0,1] \to C([0,1])$ is upper semi-continuous and $\dim(G(f)) = 1$, then $\{x \in [0,1] \mid f(x)$ is nondegenerate$\}$ is of the first category.*

Corollary 3.4. *If $f : [0,1] \to C([0,1])$ is upper semi-continuous and $\dim(G(f)) = 1$ and J is a nondegenerate subinterval of $[0,1]$, then there is an uncountable dense subset of J on which f is single-valued.*

Suppose $f : [0,1] \nearrow [0,1]$ is an upper semi-continuous set-valued function. We say that y is a *flat spot* for f provided $f^{-1}(y)$ contains a nondegenerate interval. Subintervals J_1 and J_2 of $[0,1]$ are said to be *nonoverlapping* provided if $p \in J_1 \cap J_2$, then p is an endpoint of both J_1 and J_2. Because of its separability, the interval $[0,1]$ can contain at most countably many nonoverlapping intervals. In the next two theorems we make use of the following result. A necessary and sufficient condition that a subset C of $[0,1]^2$ be 2-dimensional is that C contain a non-empty open set [3, p. 44, Theorem IV 3,].

Theorem 3.5. *If $f : [0,1] \to C([0,1])$ is upper semi-continuous and $\dim(G(f)) = 1$, then $\{y \in [0,1] \mid y$ is a flat spot for $f\}$ is at most countable.*

Proof. Suppose y_1 and y_2 are two points of $[0,1]$ with $y_1 < y_2$ and J_i is a nondegenerate interval lying in $f^{-1}(y_i)$ for $i = 1, 2$. If $J_1 \cap J_2$ is nondegenerate, $J_1 \cap J_2$ is an interval. Because f is interval-valued, $G(f)$ contains the two-cell $(J_1 \cap J_2) \times [y_1, y_2]$, a contradiction. It follows that f cannot have uncountably many flat spots. □

3.1. Compositions and dimension.

In sections 4 and 5 it is critical to our proofs concerning dimension and treelikeness of inverse limits that the dimension of compositions of the bonding functions be not greater than 1. In our next theorem we see that, for interval-valued functions on $[0,1]$, one way that we can lose control of dimension is to have a nondegenerate value for the second function in a composition at a flat spot for the first.

Theorem 3.6. *Suppose each of f and g is an upper semi-continuous function from $[0,1]$ into $C([0,1]$ such that $\dim(G(g \circ f)) = 1$. If $t \in [0,1]$ and $f^{-1}(t)$ contains an interval, then $g(t)$ is a singleton.*

Proof. Suppose $a < b$ and $[a,b] \subseteq f^{-1}(t)$, but $g(t)$ is not a singleton. Then there exist points c and d of $[0,1]$ with $c < d$ such that $g(t) = [c,d]$. Thus, $G(g \circ f)$ contains the 2-cell $[a,b] \times [c,d]$, a contradiction. □

In general, the dimension of the graphs of compositions of a sequence of interval-valued functions on $[0,1]$ can exceed the dimension of the graphs of any of the factors if some flat spot for a term of the sequence of bonding functions iterates to a point where an earlier term of the sequence has a nondegenerate value as in [6, Example 5.6]. It would be interesting to know if this is the only way a graph of such a composition can have dimension greater than 1.

4. Dimension of Inverse Limits

In this section we obtain a sufficient condition that a nondegenerate inverse limit with upper semi-continuous interval-valued bonding functions is 1-dimensional. We begin with the following observation from [7, Corollary 4.2].

Theorem 4.1. *Suppose $\boldsymbol{X}$ is a sequence of compacta and $f_i : X_{i+1} \to 2^{X_i}$ for each positive integer i. Then $\varprojlim \boldsymbol{f}$ is homeomorphic to an inverse limit on the sequence $X_1, G'_1, G'_2, G'_3, \ldots$ with bonding functions that are mappings.*

Theorem 4.1 often allows us to reduce questions about an inverse limit with set-valued functions to questions about the nature of the spaces G'_n. Properties that are preserved under inverse limits with mappings such as chainability, treelikeness, and dimension are among those that can be detected in an inverse limit with set-valued functions by studying these subsets of finite products. We have been particularly interested in determining chainability and treelikeness in such inverse limits. As can be seen in section 5, a key ingredient in such investigations is dimension. Our next theorem deals with the dimension of the spaces G'_n in inverse limits on intervals using interval-valued bonding functions.

Theorem 4.2. *Suppose n is a positive integer and $f_i : [0,1] \to C([0,1])$ is upper semi-continuous for each positive integer i, $1 \le i \le n$. Suppose further that if i and j are integers such that $1 \le i \le j \le n+1$, then $\dim(G(f_{i\,j})) = 1$. Then $\dim(G'_n) \le 1$.*

Proof. Suppose H is a nondegenerate subcontinuum of G'_n. It follows from Theorem 3.1 that if i and j are integers, $1 \le i \le j \le n+1$, then $f_{i\,j}$ is an upper semi-continuous function from $[0,1]$ into $C([0,1])$; by hypothesis, $\dim(G(f_{i\,j})) = 1$. There is a positive integer k, $1 \le k \le n+1$, such that $p_k(H)$ is a nondegenerate interval $[a,b]$. By Theorem 3.5, if j is an integer, $k \le j \le n+1$, then $\{x \in [0,1] \mid f_{k\,j}^{-1}(x)$ is of dimension $1\}$ is at most countable. If i is an integer, $1 \le i \le k$, it follows by Corollary 3.3 that $\{x \in [0,1] \mid f_{i\,k}(x)$ is nondegenerate$\}$ is of the first category. Consequently, there is a point z, $a < z < b$, such that $f_{i\,k}(z)$ is a singleton for each integer i, $1 \le i \le k$, and $f_{k\,j}^{-1}(z)$ is 0-dimensional for each integer j, $k \le j \le n+1$. Thus, $\{\boldsymbol{x} \in H \mid x_k = z\}$ is 0-dimensional and separates H; so we have that $\dim(H) = 1$. Because G'_n does not contain a 2-dimensional subcontinuum, $\dim(G'_n) \leq 1$ [3, p. 94, Theorem VI 8]. $\square$

Theorems 4.1 and 4.2 yield the theorem we sought in this section.

Theorem 4.3. *Suppose $f_i : [0,1] \to C([0,1])$ is upper semi-continuous for each positive integer i. Suppose further that if i and j are integers such that $1 \leq i \leq j$, then $\dim(G(f_{i\,j})) = 1$. Then $\dim(\varprojlim \boldsymbol{f}) \leq 1$.*

5. Treelikeness

In [5, Theorem 3.4] we showed that, under certain conditions, an upper semi-continuous function on $[0,1]$ that is the union of two mappings produces a treelike continuum. In Theorem 5.6 we extend that theorem to upper semi-continuous functions that are the union of certain finite collections of interval-valued upper semi-continuous functions on $[0,1]$. This theorem requires that the dimension of the graphs of finite compositions of elements of the collection be 1. Finite compositions of functions, each having graphs that are the union of two mappings, have 1-dimensional graphs, so Theorem 5.6 generalizes our earlier result from [5].

Włodzimierz J. Charatonik and Robert P. Roe have proved a theorem [2, Theorem 2] that we make use of in this section; we state it below for the convenience of the reader. Essential to our application of this theorem to inverse limits on $[0,1]$ here is the fact that, among continua of dimension 1, treelikeness is characterized by the property of having trivial shape.

Theorem 5.1 (Charatonik and Roe)**.** *Suppose $\boldsymbol{X}$ is a sequence of finite dimensional continua with trivial shape and $f_i : X_{i+1} \to C(X_i)$ is upper semi-continuous for each positive integer i. If $f_n(x)$ has trivial shape for each positive integer n and each $x \in X_{n+1}$, then $\varprojlim \boldsymbol{f}$ has trivial shape.*

Our next theorem extends the class of set-valued functions on $[0,1]$ known to produce treelike continua in an inverse limit. In section 6 we give an example that we show to be treelike by making use of this theorem. None of the previously known treelikeness theorems apply to Example 6.1.

Theorem 5.2. *Suppose $f_i : [0,1] \to C([0,1])$ is upper semi-continuous for each positive integer i. Suppose further that if i and j are integers such that $1 \leq i \leq j$, then $\dim(G(f_{i\,j})) = 1$. Then $\varprojlim \boldsymbol{f}$ is treelike.*

Proof. Let $M = \varprojlim \boldsymbol{f}$. By Theorem 5.1, M has trivial shape. If M is nondegenerate, by Theorem 4.3 its dimension is 1. Thus, M is treelike. $\square$

5.1. Clumps and treelikeness.

We now move to extend [5, Theorem 3.4]. As was the case in that article, our proof relies on the following theorem of H. Cook [1, Theorem 12]; in Cook's theorem, once again dimension 1 is a vital ingredient. Before stating Cook's result, for the convenience of the reader, we define some terms from his paper. A collection $\mathcal{G}$ of continua is called a *clump* provided the union of all the elements of $\mathcal{G}$, denoted $\mathcal{G}^*$, is a continuum and there is a continuum C such that C is a proper subcontinuum of each element of $\mathcal{G}$ and C is the intersection of each two elements of $\mathcal{G}$. We call a clump *usc* (Cook uses the term upper semi-continuous) provided that if $p_1, p_2, p_3, \dots$ and $q_1, q_2, q_3, \dots$ are two sequences of points of $\mathcal{G}^*$ converging to points p and q, respectively, of $\mathcal{G}^* - C$ and such that p_i and q_i belong to the same element of $\mathcal{G}$ for each $i \in \mathbb{N}$, then p and q belong to the same element of $\mathcal{G}$.

Theorem 5.3 (Cook)**.** *If $\mathcal{G}$ is a clump of treelike continua such that $\mathcal{G}$ is usc and* $\dim(\mathcal{G}^*)$*=1, then $\mathcal{G}^*$ is treelike.*

If $f, g : [0,1] \nearrow [0,1]$ are upper semi-continuous set-valued functions, we call a point $x \in [0,1]$ a *coincidence point* for f and g provided $f(x) \cap g(x) \neq \emptyset$.

Theorem 5.4. *Suppose $\mathcal{F}$ is a collection of upper semi-continuous functions on $[0,1]$ such that the union of the graphs of the elements of $\mathcal{F}$ is the graph of an upper semi-continuous function F and if $\boldsymbol{g}$ is a sequence of elements of $\mathcal{F}$, then $\varprojlim \boldsymbol{g}$ is a continuum. If there is a point $x \in [0,1]$ such that (1) if $f \in \mathcal{F}$, then $f(x) = x$ and $f^{-1}(x) = \{x\}$ and (2) if f and g are two elements of $\mathcal{F}$, then x is the only coincidence point for f and g, then $\mathcal{G} = \{\varprojlim \boldsymbol{f} \mid f_i \in \mathcal{F}$ for each positive integer $i\}$ is a clump.*

Proof. Note that $\mathcal{G}^*$, the union of the elements of $\mathcal{G}$, is $\varprojlim \boldsymbol{F}$ so $\mathcal{G}^*$ is closed. Each element of $\mathcal{G}$ is a continuum containing the point $(x, x, x, \dots)$, so $\mathcal{G}^*$ is a continuum.

Observe that if $\boldsymbol{p} \in \varprojlim \boldsymbol{F}$ and i is a positive integer such that $p_i = x$, then $\boldsymbol{p} = (x, x, x, \dots)$; this follows from the condition that $f(x) = x$ and $f^{-1}(x) = \{x\}$ for each $f \in \mathcal{F}$. Let $C = \{(x, x, x, \dots)\}$ and suppose $H = \varprojlim \boldsymbol{f}$ and $K = \varprojlim \boldsymbol{g}$ are elements of $\mathcal{G}$ and $\boldsymbol{y} \in H \cap K$. If $H \neq K$, then there is an integer i such that $f_i \neq g_i$. However, $y_i \in f_i(y_{i+1}) \cap g_i(y_{i+1})$, so $y_i = x$, and thus $\boldsymbol{y} = (x, x, x, \dots)$. Therefore, for each two elements H and K of $\mathcal{G}$, $H \cap K = C$. Thus, $\mathcal{G}^*$ is a clump. $\square$

Requiring that $\mathcal{F}$ be finite in Theorem 5.4 allows us to prove that the resulting clump is a usc clump.

Theorem 5.5. *Suppose $\mathcal{F}$ is a finite collection of upper semi-continuous functions on $[0,1]$ such that the union of the graphs of the elements of $\mathcal{F}$ is the graph of an upper semi-continuous function F, and if $\boldsymbol{g}$ is a sequence of elements of $\mathcal{F}$, then $\varprojlim \boldsymbol{g}$ is a continuum. If there is a point $x \in [0,1]$ such that (1) if $f \in \mathcal{F}$, then $f(x) = x$ and $f^{-1}(x) = \{x\}$ and (2) if f and g are two elements of $\mathcal{F}$, then x is the only coincidence point for f and g, then $\mathcal{G} = \{\varprojlim \boldsymbol{f} \mid f_i \in \mathcal{F}$ for each positive integer $i\}$ is a usc clump.*

Proof. Suppose $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}, \dots$ and $\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3}, \dots$ are sequences of points of $\mathcal{G}^*$ converging to $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively, such that $\boldsymbol{p_i}$ and $\boldsymbol{q_i}$ belong to the same element of $\mathcal{G}$ for each $i \in \mathbb{N}$ but $\boldsymbol{p} \neq (x, x, x, \dots) \neq \boldsymbol{q}$. For each $i \in \mathbb{N}$, suppose $\boldsymbol{g^i}$ is a sequence of elements of $\mathcal{F}$ such that $\boldsymbol{p_i}$ and $\boldsymbol{q_i}$ belong to $\varprojlim \boldsymbol{g^i}$. Assume $\boldsymbol{a}$ and $\boldsymbol{b}$ are sequences of elements of $\mathcal{F}$ such that $\boldsymbol{p} \in \varprojlim \boldsymbol{a}$ and $\boldsymbol{q} \in \varprojlim \boldsymbol{b}$, but $\boldsymbol{q} \notin \varprojlim \boldsymbol{a}$. There is a positive integer j such that $\pi_j(\boldsymbol{q}) \notin a_j(\pi_{j+1}(\boldsymbol{q}))$. For this integer j, consider the sequence $g_j^1, g_j^2, g_j^3, \dots$. Because $\mathcal{F}$ is finite, there is an element h of $\mathcal{F}$ such that $g_j^i = h$ for infinitely many integers i. Because $\pi_j(\boldsymbol{p_i}) \in g_j^i(\pi_{j+1}(\boldsymbol{p_i}))$ and $\pi_j(\boldsymbol{q_i}) \in g_j^i(\pi_{j+1}(\boldsymbol{q_i}))$ for each $i \in \mathbb{N}$, it follows that $\pi_j(\boldsymbol{p}) \in h(\pi_{j+1}(\boldsymbol{p}))$ and $\pi_j(\boldsymbol{q}) \in h(\pi_{j+1}(\boldsymbol{q}))$. Because $\pi_j(\boldsymbol{q}) \notin a_j(\pi_{j+1}(\boldsymbol{q}))$, $h \neq a_j$. However, $\pi_j(\boldsymbol{p}) \in a_j(\pi_{j+1}(\boldsymbol{p}))$ and $\pi_j(\boldsymbol{p}) \in h(\pi_{j+1}(\boldsymbol{p}))$; thus, $\pi_{j+1}(\boldsymbol{p}) = x$. Therefore, $\boldsymbol{p} = (x, x, x, \dots)$, a contradiction. Consequently, $\boldsymbol{p}$ and $\boldsymbol{q}$ belong to the same element of $\mathcal{G}$, so $\mathcal{G}$ is a usc clump. □

If we now add that the elements of a finite collection $\mathcal{F}$ are interval-valued with 1-dimensional graphs and all finite compositions of elements of $\mathcal{F}$ have 1-dimensional graphs, we obtain the theorem we seek generalizing [5, Theorem 3.4].

Theorem 5.6. *Suppose $\mathcal{F}$ is a finite collection of upper semi-continuous interval-valued functions on $[0,1]$ with 1-dimensional graphs such that the union of the graphs of the elements of $\mathcal{F}$ is the graph of an upper semi-continuous function F. Suppose further that if $f_1, f_2, \dots, f_n$ is a finite sequence of elements of $\mathcal{F}$ and i and j are integers such that $1 \leq i \leq j \leq n+1$, then $\dim(G(f_{i\,j})) = 1$. If there is a point $x \in [0,1]$ such that (1) if $f \in \mathcal{F}$, then $f(x) = x$ and $f^{-1}(x) = \{x\}$ and (2) if f and g are two elements of $\mathcal{F}$, then x is the only coincidence point for f and g, then $\varprojlim \boldsymbol{F}$ is a treelike continuum.*

Proof. Let $M = \varprojlim \boldsymbol{F}$ and $\mathcal{G} = \{\varprojlim \boldsymbol{f} \mid f_i \in \mathcal{F}$ for each positive integer $i\}$. Observe that $M = \mathcal{G}^*$. We may assume that M is nondegenerate. Because each element of $\mathcal{G}$ is an inverse limit with interval-valued functions, $\mathcal{G}$ is a collection of continua. By Theorem 5.5, $\mathcal{G}$ is a usc clump. By Theorem 5.2, each element of $\mathcal{G}$ is treelike. Thus, if M is 1-dimensional, then M is treelike.

To see that M is 1-dimensional we show that G'_n is 1-dimensional for each positive integer n and apply Theorem 4.1. Note that if $n \in \mathbb{N}$, then $G'_n = \mathcal{C}^*$ where $\mathcal{C} = \{G'(f_1, f_2, \dots, f_n) \mid f_i \in \mathcal{F}$ for each i, $1 \leq i \leq n\}$. Each element of $\mathcal{C}$ is a continuum and each element of $\mathcal{C}$ is 1-dimensional by Theorem 4.2. Because $\mathcal{F}$ is finite, $\mathcal{C}$ is finite, and thus $\mathcal{C}^*$ is 1-dimensional. □

Examples 4.2 and 4.3 in [5] show that, even in the case that $\mathcal{F}$ consists of two mappings, in order to prove treelikeness in Theorem 5.6, some conditions are needed such as our hypothesis requiring the existence of the point $x \in [0,1]$ satisfying the two conditions (1) if $f \in \mathcal{F}$, then $f(x) = x$ and $f^{-1}(x) = \{x\}$ and (2) if f and g are two elements of $\mathcal{F}$, then x is the only coincidence point for f and g.

6. Example

There are several known examples of interval-valued functions on $[0,1]$ having both flat spots and nondegenerate values that nonetheless produce treelike continua. These include [4, Example 2.4, Example 2.6, and Example 2.23] although each of these has only finitely many flat spots and finitely many nondegenerate values. Example 5.3 in [6] showed that a treelike continuum can be produced in an inverse limit with a single bonding function with infinitely many flat spots. In this section we give an example of an interval-valued bonding function that has both flat spots and infinitely many nondegenerate values while still producing a treelike inverse limit. Because the closure of the set of nondegenerate values is the entire interval, none of the previously known results (including [6, Theorem 4.2]) apply.

Example 6.1. Let $D = \{1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, \dots\}$, i.e., D is the set of dyadic rationals in $[0,1]$. Let $f : [0,1] \to C([0,1]$ be given by $f(t) = [0, 1/2^{q-1}]$ where $t = p/2^q$ with p odd and less than 2^q and $f(t) = 0$ otherwise (see Figure 1 for the graph of f). Then $\varprojlim \boldsymbol{f}$ is a treelike continuum.

Proof. Suppose $n \in \mathbb{N}$; we show that $\dim(G(f^n)) = 1$. To see this, suppose x is a point of $[0,1]$ that is not a dyadic rational. Then $f(x) = 0$. Because $f(0) = 0$, $\{(x,0)\}$ is the only point of $G(f^n)$ having first coordinate x. It follows that $G(f^n)$ does not contain a 2-cell and, consequently, is 1-dimensional. By Theorem 5.2, $\varprojlim \boldsymbol{f}$ is treelike. □

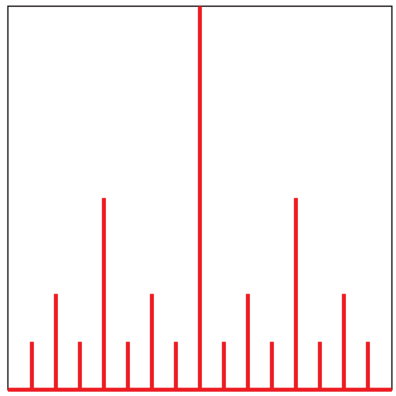

FIGURE 1. A depiction of the graph of the bonding function in Example 6.1.

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