

APPLICATIONS OF PURE MATHEMATICS

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ABSTRACT. Many mathematicians pursue their art out of curiosity, not because they have some predetermined application in mind. That does not necessarily mean that they are not interested in applications to “real world” problems—it is just not their primary motivation. In this talk intended for a campus-wide audience of students and faculty I will discuss a research topic in my field of mathematics, topology, that has, to my surprise, been found to have applications outside mathematics.

1. INTRODUCTION

Applications of mathematics are sometimes found in unexpected places. Pure mathematicians are seldom, if ever, motivated by application of their work. Some even go so far as to say they do not expect their work ever to be applied. I have heard it attributed to G. H. Hardy that he particularly liked one of his theorems simply because, in his opinion, it would never have any practical use. If you saw the movie, *The Man Who Knew Infinity*, you are aware that it featured Hardy and his relationship with Ramanujan. I could not confirm that Hardy actually said that, but it is my understanding that some of his work on prime factorization comprises the basis for modern methods of secure electronic communication.

What I could find was the following quote from Hardy’s essay, *A mathematician’s apology*, which I highly recommend that you read sometime. I easily located it online using Google.

"I have never done anything ‘useful’. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."

Unlike Hardy, I have always harbored the hope that, some day, perhaps not in my lifetime, some of the things I worked on would be used to help solve problems outside mathematics.

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Today, I will talk of potential applications of some of my work in the field of economics.

2. TOPOLOGICAL EQUIVALENCE

My area of mathematics is topology. I have heard it said that a topologist is a mathematician who cannot tell the difference between a donut and a coffee cup. Now, I can assure everyone that, so far, I have never bitten into my coffee cup thinking it was a donut nor have I tried to pour my morning coffee into my donut. However, there is a sense in which that statement is true.

Imagine for a moment that you take a lump of Play-Doh. Roll it out into a long tubular shape and join the ends to form our 'donut'. Now, hold this 'donut' up on its side, and without tearing or breaking it, make a small indentation in it. Continue pushing and enlarging the indentation until it begins to take a shape that could hold coffee. With some more adjustments, you may now have in hand a crude coffee cup. In case you had trouble following my directions in your head, see Figure 1 for a picture of the process I just described. It is taken from the Life Science Library book, *Mathematics*, published in 1963. What I have described is a topological transformation from the 'donut' to the 'coffee cup'. These objects, being topologically equivalent, form the basis of the statement about topologists.

Two things that are not topologically equivalent are the straight line interval and the circle. One reason for this is the following. Imagine a piece of string (the interval) and a rubber band (the circle). No matter where you cut the rubber band with one cut it remains in one piece. Unless you somehow 'magically' cut the string at the end, however, the result is two pieces.

Now, for an illustration of similar looking objects that are not topologically equivalent. Imagine taking a long strip of paper and gluing the ends together to form a cylindrical object. Again, taking a long strip of paper, this time we glue the ends together but in doing so we introduce a twist. These two objects so obtained are not topologically equivalent. One way to see a difference is as follows. Take a pair of scissors and start cutting the first object, the cylinder, along its center. When we get all the way around, we wind up with two pieces. With the second object, such a cut leaves the object in one piece. Interestingly, the twist gives us a means of making a mark on both sides of a strip of paper without removing the pencil from the paper. This second object with the twist is sometimes referred to as a Möebius strip.

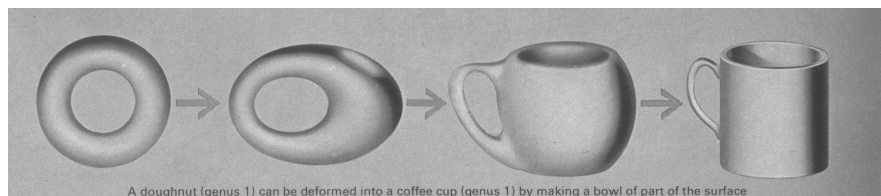


FIGURE 1. Morphing a donut into a coffee cup.

3. DYNAMICS

Topology is the study of transformations of all sorts. When we molded the Play-Doh from the torus into the coffee cup, we were making a topological transformation. Before that, when we took the Play-Doh and stuck its ends together, we were making a continuous transformation as we were with the strip of paper when we glued the ends together, with or without the twist.

Generally, a dynamical system consists of some object and a continuous transformation of that object into itself. In a dynamical system, one construction of particular interest is the following: choose some starting point and follow its progress under the transformation as we apply the transformation over and over. If this settles into some sort of pattern after many iterations, the pattern is often referred to as an attractor in the system. This process becomes especially interesting if one uses a parameterized family of transformations. The biologist, Robert May, studying population dynamics in the 70s called particular attention to the parameterized logistic family of transformations of the interval $[0, 1]$, namely, $f_\lambda(x) = 4\lambda x(1 - x)$ where the parameter $\lambda \in [0, 1]$. This was followed in the 80s by a study of the behavior of this system by Feigenbaum employing a hand-held calculator. Computer generated pictures of the attractors soon became popular in so-called bifurcation diagrams (see Figure 2 for the diagram for the logistic family). Buzz words like CHAOS served to draw additional attention to dynamical systems.

About the same time Benoit Mandelbrot generated colorful attractive pictures related to iteration of the origin by the simple parameterized family of functions of a complex variable, $f(z) = z^2 + c$. See Figure 3 for an illustration of the Mandelbrot set.

Long before these events that were made possible by computing machinery, however, mathematicians like Sarkovskii, Fatou, and Julia laid

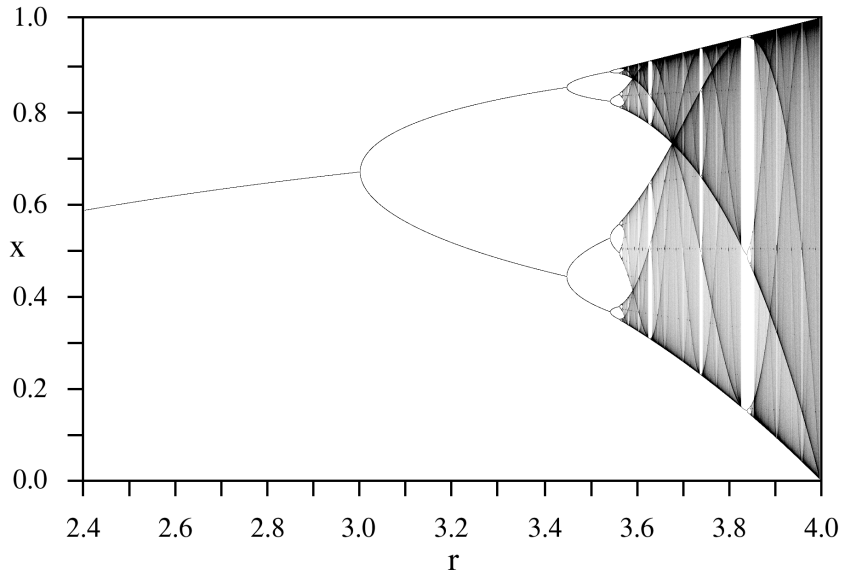


FIGURE 2. A bifurcation diagram for the logistic family.

the theoretical groundwork that explained the emerging computer graphics.

4. INVERSE LIMITS

I would like to turn now to a construction that makes use of an object and a continuous transformation of the object into itself. We just mentioned dynamical systems and the dynamicist's interest in following iterations under the transformation. However, instead of following someplace in the object under the transformation, let us instead look backward to see where our starting place came from. Now, there could be two or even more choices (as with the spot where we glued the strip of paper together). If there is a choice, we make a choice and repeat the process. If we continue this process, we could end up with an infinite sequence of places inside the object we started with. To make this interesting topologically, we decide that two such sequences are close together if they are very close for many, many steps of the process; and we consider all possible such sequences.

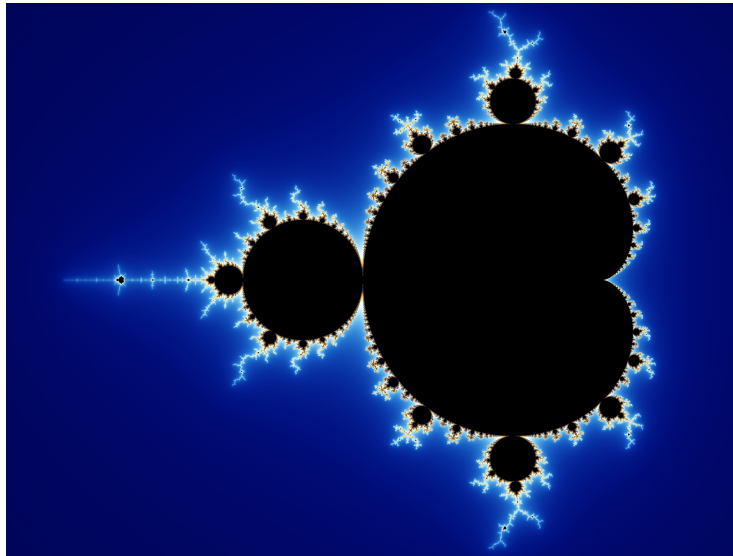
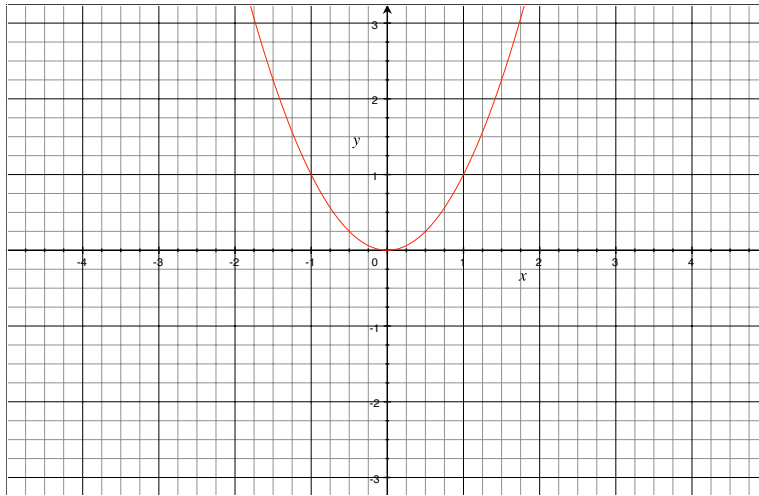
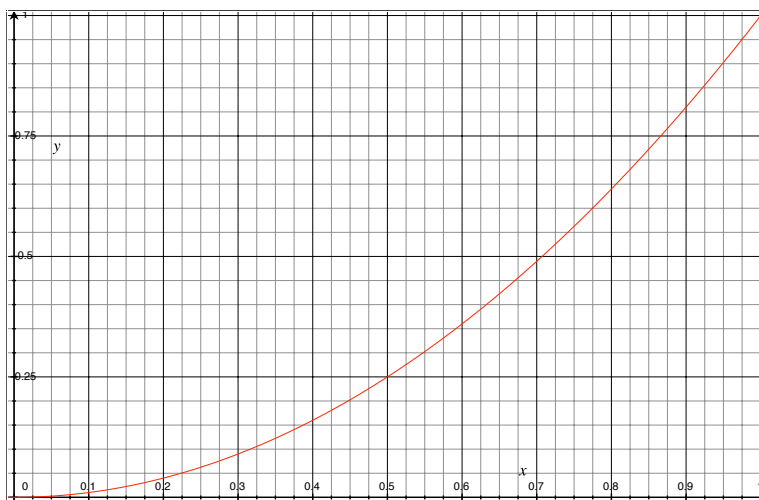


FIGURE 3. The Mandelbrot set.

To get a feeling for this process, let's consider a couple of simple functions. Both are simple parabolas. The first one is $y = x^2$. Its graph is shown in Figure 4. We restrict our attention to the interval $[0, 1]$. That graph is shown in Figure 5. Select some number in the interval and look back to see where it came from. If 0 is our choice, it came from 0. That, of course, came from 0, and so forth; so the sequence starting with 0 is $0, 0, 0, \dots$. Similarly, if our starting point is 1, it came from 1, etc. so this time the sequence is $1, 1, 1, \dots$. Starting from $1/2$, this came from $1/\sqrt{2}$, which came from $1/\sqrt{\sqrt{2}}, \dots$. Notice, no matter where we start, at every stage there is only one choice of where it came from; so every sequence is uniquely determined by its starting point. Thus, the collection of sequences is pretty much like the interval itself; i.e., the collection of sequences is topologically equivalent to the interval. Interesting, but not too interesting.

FIGURE 4. The parabola $y = x^2$.

Let's take a different parabola: $y = 4x(1 - x)$. This parabola is one in that parameterized family of parabolas studied in population dynamics suggested by the biologist Robert May back in the 1970s. Its graph is shown in Figure 6 and on $[0, 1]$ in Figure 7. As before, we restrict our attention to $[0, 1]$. Now, starting with 1, notice that it comes from $1/2$; but $1/2$ comes from two possibilities. We choose one of the possibilities, but looking back it comes from two possibilities. In fact, no matter where we start, we quickly are faced with a choice for the next term of the sequence. One place we could start is at 0; doing so gives us 0 and 1 as the choices of where it came from. By choosing 0, and continuing to make this choice, we see that $0, 0, 0, \dots$ is a sequence in the collection of sequences. But, very close by to this sequence is one that starts out with 0s for a long time but then we opt to choose 1 which then forces $1/2$, and then gets us back to two choices, etc. The collection of all sequences we obtain seems to be very complex. In fact, it is! Figure 8 is a picture depicting something that is topologically equivalent to the collection of sequences—i.e., the ‘coffee cup’ for our ‘donut’ of sequences.

FIGURE 5. The parabola $y = x^2$ on $[0, 1]$.

This thing is quite interesting. It contains the famous Smale horseshoe, an attractor in a dynamical system made famous in the 1960s by Steven Smale. But, it was studied much earlier, almost 100 years ago, by a Polish mathematician, Janiszewski (and subsequently by the well-known Polish mathematician Knaster who first drew this picture of it). Both were trying to understand what was at the time a recent development by L. E. J. Brouwer who in 1910 produced the first example of something now known as an indecomposable continuum—a compact, connected set that is not the union of two proper compact, connected subsets. This is not easy to imagine even existing—most of the objects we normally encounter are not like this. When we took the scissors to the cylinder earlier, in essence, we were getting it to be the union of two proper subcontinua.

What just happened? We began with a simple parabola and with the construction of this collection of sequences we produced something that was totally unknown just a little over 100 years ago! Maybe it is a bit easier to understand why someone might find it intellectually stimulating to study this type of construction in more detail. This construction goes by

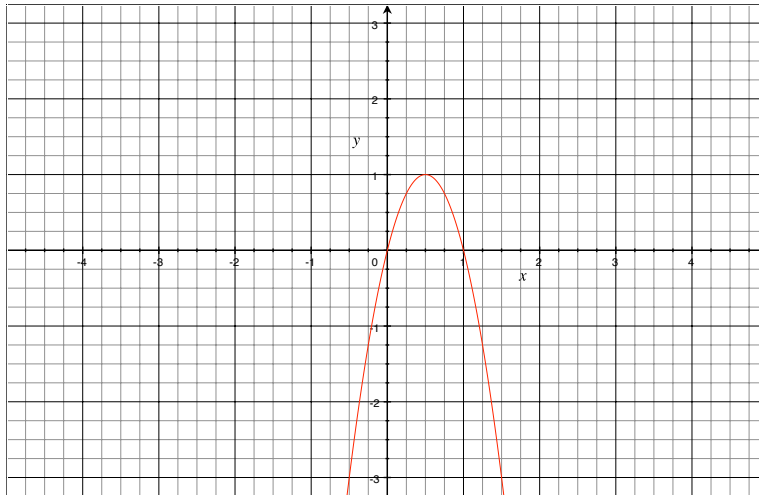


FIGURE 6. The parabola $y = 4\lambda x(1 - x)$.

the name of inverse limit. It has proven to be invaluable as a tool for constructing complicated spaces from simple objects. I spent a lot of the early part of my career making use of the construction to build complicated objects that answered several questions that had eluded mathematicians in continuum theory for a number of years. Many others employed the construction to answer other continuum theoretic questions. I am pleased to note that it is still being found of value. These “applications” of the inverse limit construction in mathematics helped prompt me, along with Bill Mahavier of Emory University, to gather a lot of the techniques of inverse limits into a Springer book after my retirement here at Rolla.

5. APPLICATIONS

But, what about applications of inverse limits outside of mathematics? People working in economics have found that the inverse limit construction is perfect for one way they have of studying their economic models. A model in economics is in its simplest form some set and a function on that set used to describe the change from the present economic situation

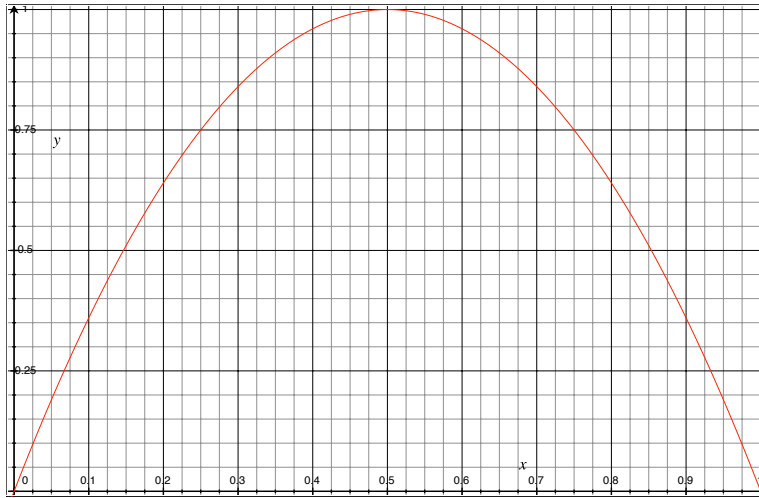


FIGURE 7. The parabola $y = 4\lambda x(1 - x)$ on $[0, 1]$.

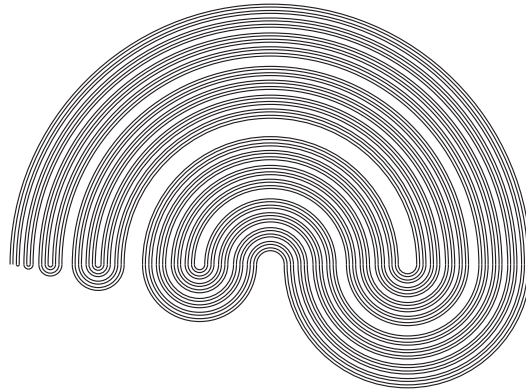


FIGURE 8. An indecomposable continuum

to a future one. Does that sound familiar? The interpretation is perhaps different but it is still a dynamical system. Sometimes, however, the relationship between the present state and the future state is not expressed by a function (single-valued). Instead, there can be multiple potential outcomes. In some cases, though, some future state gives rise to a single present state. Economists describe this phenomenon as “backward economics”, and their interest is expressed by sequences of states each term of which determines a unique previous term. Looking at all such sequences is exactly the set-up for an inverse limit. Mathematicians working with economists have written about this very situation. Two such are papers by Raines and Medio and by Kennedy, Stockman, and Yorke. Raines was actually one of my master’s degree students here who went on to earn a D.Phil. at Oxford. If you are interested, I refer you to these papers as a starting point for diving deeper into the application of the pure mathematical construction of an inverse limit into the field of economics.

Alfredo Medio and Brian Raines, *Inverse limit spaces arising from problems in economics*, *Topology and Its Applications* **153** (2006), 3437–3449.

Judy Kennedy, David R. Stockman, and James A. Yorke, *Inverse limits and an implicitly defined difference equation from economics*, *Topology and Its Applications* **154** (2007), 2533–2552.

In conclusion, one never knows where the next application of mathematics may arise. Ideas pursued by a mathematician out of his or her own curiosity may turn out to have far reaching applications far from the mind of the person first looking into them. There was a phrase that cropped up in the 70s and 80s—“applicable mathematics”. I always thought it was pompous for anyone to claim they knew which mathematics would ever be applied and which (by inference) would not, even if the mathematician like Hardy thought something they produced would never be useful.

To sum up then my message is a simple one: curiosity may have killed the cat but it motivates the mathematician as I am confident that it does researchers across this campus.

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