ATRIODIC TREE-LIKE CONTINUA: PROBLEMS AND RECENT RESULTS

W. T. Ingram

1. Introduction

In the last few years research activity has expanded rapidly in the area of atriodic tree-like continua. The existence of such continua which are not chainable was demonstrated in [3], and the existence of such without the fixed-point property has been demonstrated in the last couple of years [1]. Answers to other intriguing questions such as Lelek's "Does span zero imply chainability?" definitely lie in the study of atriodic tree-like continua.

Throughout this paper a continuum is a compact, connected metric space. A continuum is a triod if some subcontinuum of it separates it into three mutually separated sets. A continuum is atriodic provided it contains no triod. A continuum has span zero provided each subcontinuum K of $M \times M$ such that $p_1K = p_2K$ intersects the diagonal.

2. Weak Chainability

In [5] Lelek defined weak chainability of continua and characterized it by the property of being a continuous image of a chainable continuum. In his doctoral dissertation at the University of Houston Travis Moebes [7] proved that the continuum in [3] is not weakly chainable. His proof is quite combinatorial in nature but it may be useful in constructing non-weakly chainable continua. In particular, perhaps using his techniques one could obtain a non-weakly chainable continuum with span zero answering:

Question 1. Is there a continuum with span zero which is not weakly chainable?

3. Plane Continua

Still unanswered is the famous old question of whether all non-separating plane continua have the fixed-point property. Recently, Brechner and Mayer [2] have proved that the example of [3] has the fixed-point property. In a separate paper Mayer [6] proposes an example of a planar atriodic tree-like continuum which does not fit their proof and it could possibly fail to have the fixed-point property.

At the Spring Topology Conference at Auburn University in 1976 the author asked if there is a non-planar atriodic tree-like continuum. In an as yet unpublished paper Oversteegen and Tymchatyn [8] have constructed an example of an atriodic tree-like continuum which cannot be embedded in the plane. The heart of their construction involves modifying the example of [3] to obtain a continuum which does not have property U (a continuum M has property U provided there exists in the plane uncountably many mutually exclusive copies of M). This part of their construction reminds us of the question raised by the author here in 1977 [4]:

Question 2. Is there an atriodic tree-like continuum with positive span which has property U?

Further, Oversteeger and Tymchatyn proved that each non-separating planar homogeneous continuum has span zero. This partially answers another question the author raised here in 1977.

Question 3. Is there a homogeneous, tree-like continuum with positive span?

We conclude this section with a question related to Question 2. The sequence $M_1, M_2, M_3, \ldots$ is said to converge homeomorphically to the continuum M provided for each positive integer i there exists a homeomorphism h_i throwing M_i onto M such that $d(h_i(x), x) < 1/i$ for each x in M_i.

Question 4. If $M_1, M_2, M_3, \ldots$ is a sequence of mutually exclusive plane continua converging homeomorphically to the non-separating plane continuum M, is the span of M zero?

4. Span

If f is a mapping of the continuum X onto the continuum Y, the span of f, denoted σf, is the least upper bound of the set to which the non-negative number ε belongs if and only if there exists a continuum Z in $X \times X$ such that $p_1 Z = p_2 Z$ and $d(f(x), f(y)) \geq \varepsilon$ for each (x, y) in Z. The span of X, denoted σX, is the span of the identity on X.

In [3] the author proved: If X is the inverse limit of the inverse limit sequence $\{X_i, f_i^j\}$, $\varepsilon > 0$, and there is a positive integer n such that if $j > n$ then $\sigma f_n^j \geq \varepsilon$, then $\sigma X > 0$.

We conclude this paper with a proof of a theorem converse to the one stated above. Although this theorem is not in print (insofar as the author knows) the author has had some advanced topology students prove the theorem in studying span.

Theorem. Suppose $X = \varprojlim \{x_i, f_i^j\}$ has positive span. Then there are a positive number η and a positive integer i such that if $j \geq i$ then $\sigma f_i^j \geq \eta$.

Proof. Suppose $\sigma X = \varepsilon > 0$ and Z is a continuum in $X \times X$ such that if (x, y) is in Z then $d(x, y) \geq \varepsilon$. (We consider X metrized by $d(x, y) = \sum_{i \geq 1} \frac{d_i(x_i, y_i)}{2^i}$ where each factor X_i has diameter 1.) Assume that for each i and $\delta > 0$ there exists a positive integer N such that if $j \geq N$ then $\sigma f_i^j < \delta$. Let i be a positive integer such that $\sum_{j \geq i} \frac{1}{2^j} \leq \frac{\varepsilon}{2}$. There is a positive number δ such that if p and q are in X_i and $d_i(p, q) < \delta$ then for each $k \leq i$, $d_k(f_k^i(p), f_k^i(q)) < \varepsilon/2$. Suppose j is an integer greater than i so that $\sigma f_i^j < \delta$.

Let $Z_j = (\pi_j \times \pi_j)(Z)$. We show that $p_1 Z_j = p_2 Z_j$. If x is in $p_1 Z_j$ then there is a point (Z^1, Z^2) of Z such that $x = Z_j^1$. But Z^1 is the second coordinate of some point of Z thus x is in $p_2 Z_j$. Similarly, $p_2 Z$ is a subset of $p_1 Z$.

Since $\sigma f_i^j < \delta$ there is a point (x, y) of Z_j such that $d_i(f_i^j(x), f_i^j(y)) < \delta$ and a point (Z^1, Z^2) of Z such that $(\pi_j \times \pi_j)(Z^1, Z^2) = (x, y)$. Thus, for $k \leq i$, $d_k(Z_k^1, Z_k^2) < \varepsilon/2$ and $d(Z^1, Z^2) < \sum_{n=1}^{i} \frac{\varepsilon}{2^{n+1}} + \sum_{n=i+1}^{\infty} \frac{1}{2^n} = \varepsilon$. This is a contradiction.

REFERENCES

1. David Bellamy, A tree-like continuum without the fixed-point property, to appear.

2. B. Brechner and J. C. Mayer, The prime end structure of indecomposable continua and the fixed point property, to appear in Proceedings of Conference on Topology and Set Theory, University of California, Riverside, May, 1980.

3. W. T. Ingram, An atriodic tree-like continuum with positive span, Fund. Math. 77 (1972), 99-107.

4. __________, Tree-like continua and span, Proceedings of the Eighth Annual USL Mathematics Conference, 1977, 62-66.

5. A Lelek, On weakly chainable continua, Fund. Math., 50 (1962), 271-282.

6. John C. Mayer, Principal embeddings of atriodic plane continua, to appear Proceedings of Topology Conference, The University of Texas at Austin, June, 1980.

7. Travis Moebes, An atriodic tree-like continuum that is not weakly chainable, Doctoral Dissertation, University of Houston, August, 1980.

8. Lex G. Oversteegen and E. D. Tymchatyn, Plane strips and the span of continua, to appear.