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CONCERNING NONCONNECTED INVERSE LIMITS WITH UPPER SEMI-CONTINUOUS SET-VALUED FUNCTIONS

W. T. INGRAM

Abstract. In this paper we present a sequence $f_2, f_3, f_4, \ldots$ of upper semi-continuous set-valued functions with the property that the graph of f_n^n is not connected but the graph of f_n^k is connected for $1 \leq k < n$. Thus, for each positive integer n, $\varprojlim \boldsymbol{f_n}$ is not connected but the difficulty of detecting this fact increases with n.

1. Introduction

Inverse limits with upper semi-continuous set-valued functions are increasingly being studied. Continuum theorists are particularly interested in the question of when such inverse limits are connected. For inverse limits with mappings, this is always the case when the factor spaces are continua. With upper semi-continuous set-valued functions, connectedness of the inverse limit can easily fail. For example, the inverse limit on $[0,1]$ using the single bonding function f given by $f(x) = \{0,1\}$ for each $x \in [0,1]$ is the Cantor set. Moreover, the inverse limit can fail to be connected even if the graph of the function is connected (see Example 3.3 or [5, Example 4]). In this paper, we present some additional examples in the form of a sequence of set-valued functions with connected graphs having inverse limits that are not connected.

2. Basic definitions and preliminary results

If X is a topological space, we use 2^X to denote the collection of all closed subsets of X and $C(X)$ to denote the collection of closed and

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connected subsets of X. If each of X and Y is a topological space, a set-valued function $f : X \to 2^Y$ is called *upper semi-continuous* provided that if O is an open subset of Y that contains $f(x)$, then there is an open subset U of X containing x such that if $t \in U$, then $f(t) \subseteq O$. If $H \subseteq X$, by $f(H)$ we mean $\{y \in Y \mid \text{ there is a point } x \in X \text{ such that } y \in f(x)\}$ and f is said to be *surjective* provided $f(X) = Y$. By a *Hausdorff continuum* we mean a compact, connected Hausdorff space, whereas by a *continuum* we mean a compact, connected metric space. If $X_1, X_2, X_3, \ldots$ is a sequence of topological spaces, we denote the product of the sequence by $\prod_{i>0} X_i$ and endow it with the product topology. We adopt the convention of denoting sequences (even finite sequences) in boldface type and the terms of sequence in italics. Thus, if $\boldsymbol{x}$ is a sequence, we denote its terms by $x_1, x_2, x_3, \ldots$. Because the points of $\prod_{i>0} X_i$ are sequences, it should be permissible to write such a point as $\boldsymbol{x} = x_1, x_2, x_3, \ldots$ where $x_i \in X_i$ for each positive integer i, but we adopt the convention of denoting points of the product by enclosing them in parentheses as $\boldsymbol{x} = (x_1, x_2, x_3, \ldots)$. If $A \subseteq \{1, 2, 3, \ldots\}$, we denote by $\pi_A : \prod_{i>0} X_i \twoheadrightarrow \prod_{i\in A} X_i$ the function given by $\pi_A(\boldsymbol{x}) = \boldsymbol{y}$ where $y_i = x_i$ for each $i \in A$. For each set A of positive integers, π_A is a continuous function, i.e., a *mapping*. In case $A = \{n\}$, we denote $\pi_{\{n\}}$ by π_n. If $B \subseteq X \times Y$, then $B^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in B\}$.

Suppose $X_1, X_2, X_3, \ldots$ is a sequence of compact Hausdorff spaces and, for each positive integer i, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function. By the *inverse limit* of the inverse sequence $\{X_i, f_i\}$, denoted $\varprojlim\{X_i, f_i\}$ or normally $\varprojlim \boldsymbol{f}$, we mean $\{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1}) \text{ for each positive integer } i\}$. It is known that $\varprojlim \boldsymbol{f}$ is nonempty if each bonding function f_i is upper semi-continuous [3]. If $f : X \to 2^Y$ is a set-valued function, the graph of f, denoted $G(f)$, is $\{(x, y) \in X \times Y \mid y \in f(x)\}$. It is known that if X and Y are compact Hausdorff spaces, then $f : X \to 2^Y$ is upper semi-continuous if and only if $G(f)$ is closed [3]. A sequence of subsets of $\prod_{i>0} X_i$, the terms of which approximate the inverse limit, is useful in the proof that the inverse limit is nonempty as well as in connectedness arguments. We use these sets in this article as well. If n is a positive integer, let $G_n = \{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \le i \le n\}$. The set $\pi_{\{1,2\}}(G_1) = G(f_1)^{-1}$. If $f : X \to 2^Y$ and $g : Y \to 2^Z$ are set-valued functions, the composition $g \circ f : X \to 2^Z$ is given by $(x, z) \in g \circ f$ if and only if there is a point $y \in Y$ such that $y \in f(x)$ and $z \in g(y)$. For convenience, we often denote $g \circ f$ by gf. If $f : X \to 2^X$, we denote $f \circ f$ by f^2 and, for $n > 2$, $f^n = f^{n-1} \circ f$. If $\boldsymbol{X}$ is a sequence of compact Hausdorff spaces and $\boldsymbol{f}$ is a sequence of upper semi-continuous functions such that, for each positive integer i, $f_i : X_{i+1} \to 2^{X_i}$ and j is a positive integer greater than i, we define $f_{i\,j} : X_j \to 2^{X_i}$ by $f_{i\,j} = f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$. If $f : X \to 2^Y$ is an upper semi-continuous function and A is a subset of X, by the *restriction of* f *to* A, denoted $f|A$, is meant $\{(x, y) \in f \mid x \in A\}$.

The following theorem appears in [3, Theorem 4.8]. Variations of this theorem allowing such hypotheses as $f_i^{-1} : X_i \to C(X_{i+1})$ for each $i \in \{k \mid f_k$ is not Hausdorff continuum-valued$\}$ that yield the same conclusion (see [3, Theorem 4.9]), as well as extensions to inverse limit systems over directed sets (see [4]) have been proved.

Theorem 2.1. *If $\boldsymbol{X}$ is a sequence of Hausdorff continua and $\boldsymbol{f}$ is a sequence of upper semi-continuous set-valued functions such that for each positive integer i, $f_i : X_{i+1} \to C(X_i)$, then $\varprojlim \boldsymbol{f}$ is a Hausdorff continuum.*

In [3, p. 121], it was observed that in an inverse limit sequence with surjective upper semi-continuous bonding functions $f_1, f_2, f_3, \ldots$, if $t \in X_1$, there is a point $\boldsymbol{x} \in \varprojlim \boldsymbol{f}$ such that $x_1 = t$. The following theorem is almost as easy to prove. Its proof is left to the reader.

Theorem 2.2. *Suppose $\boldsymbol{X}$ is a sequence of compact Hausdorff spaces and $f_k : X_{k+1} \to 2^{X_k}$ is upper semi-continuous and surjective for each positive integer k. If $i < j$ and $t \in f_{i\,j}(s)$, then there is a point $\boldsymbol{x} \in \varprojlim \boldsymbol{f}$ such that $x_i = t$ and $x_j = s$.*

The condition in Theorem 2.1 that the bonding functions be Hausdorff continuum-valued easily yields that $G(f_i)$ is connected for each positive integer i. Because a subset B of a product $X \times Y$ is connected if and only if B^{-1} is connected, that $G(f_i)$ is connected is also a consequence of the connectedness of the inverse limit with surjective bonding functions because $\pi_{\{i,i+1\}}(\varprojlim \boldsymbol{f}) = G(f_i^{-1}) = (G(f_i))^{-1}$. More generally, we have the following observation (see [5, Example 4] where it was nicely employed) which follows from the fact that $\pi_{\{i,j\}}$ is continuous for each two integers i and j with $i < j$.

Theorem 2.3 (Nall)**.** *If $\boldsymbol{X}$ is a sequence of compact Hausdorff spaces and $\boldsymbol{f}$ is a sequence of surjective upper semi-continuous functions such that $f_k : X_{k+1} \to 2^{X_k}$ for each positive integer k and $\varprojlim \boldsymbol{f}$ is connected, then $G(f_{i\,j})$ is connected for each two integers i and j with $i < j$.*

Proof. Let $M = \varprojlim \boldsymbol{f}$. Suppose i and j are positive integers, $1 \le i < j$. It is not difficult to establish that if $(x, y) \in \pi_{\{i,j\}}(M)$, then $x \in f_{i\,j}(y)$. Moreover, by Theorem 2.2, if $x \in f_{i\,j}(y)$, there is a point of M having ith coordinate x and jth coordinate y. It thus follows that $(G(f_{i\,j}))^{-1} = \pi_{\{i,j\}}(M)$, so $G(f_{i\,j})$ is connected. $\square$

The converse of Theorem 2.3 is not true. Examples showing this include [5, Example 4], as well as Example 3.2 and Example 3.3 in §3 of this article. Furthermore, the hypothesis that the bonding functions be surjective is necessary for if $f : [0,1] \to 2^{[0,1]}$ is the function such that $f(t) = t/3$ for $0 \le t < 1$ and $f(1) = \{1/3, 2/3\}$, then $\varprojlim \boldsymbol{f} = \{(0,0,0,\dots)\}$ is connected even though $G(f)$ is not connected.

3. Nonconnected Inverse Limits

The proof of Theorem 2.1 in [3] is achieved by showing that $G_n = \{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for $1 \le i \le n\}$ is a Hausdorff continuum for each positive integer n. From this it follows that the inverse limit is a Hausdorff continuum, being the intersection of a nested sequence of Hausdorff continua. Indeed, the condition that G_n is a Hausdorff continuum for each n is equivalent to the connectedness of the inverse limit. Moreover, $\varprojlim \boldsymbol{f}$ is a Hausdorff continuum if and only if $G'_n = \pi_{\{1,2,\dots,n+1\}}(G_n)$ is connected for each positive integer n. Unfortunately, it is often not very easy to check that either G_n or G'_n is connected.

For ordinary inverse limits with mappings, we are often able to glean information about the inverse limit by examining composites of the bonding maps. Such techniques have proven less informative for inverse limits with set-valued functions. However, for inverse limits with a single surjective upper semi-continuous bonding function f, Theorem 2.3 provides a reasonably simple way to detect that $\varprojlim \boldsymbol{f}$ is *not connected* by looking at the graphs of $f, f^2, f^3, \dots$ until we find an integer n such that $G(f^n)$ is not connected. In [5, Example 4], Van Nall presents a nice example of a surjective upper semi-continuous function $f : [0,1] \to 2^{[0,1]}$ such that $G(f)$ is connected but $G(f^2)$ is not connected. Nall's function is the union of two mappings $\varphi : [0,1] \to [0,1]$ and $\psi : [1/2,1] \twoheadrightarrow [0,1]$ where $\varphi(x) = x/2$ and $\psi(x) = 2x-1$ (in Nall's paper, φ was denoted by f_1 and ψ by f_2). For Nall's function f, it follows from Theorem 2.3 that $\varprojlim \boldsymbol{f}$ is not connected (in fact, this is the essence of Nall's proof that the inverse limit is not connected). Nall's example caused the author to ask if there is an upper semi-continuous function $f : [0,1] \to 2^{[0,1]}$ such that $G(f)$ and $G(f^2)$ are both connected but $G(f^3)$ is not connected. The function f_3 of Example 3.2 below is such a function. In fact, in Example 3.2, we present a sequence $f_2, f_3, f_4, \dots$ of upper semi-continuous functions, each term of which is a function from $[0,1]$ into $2^{[0,1]}$ such that, if n is a positive integer, then $G(f_n^k)$ is connected for $1 \le k < n$ but $G(f_n^n)$ is not connected.

In [3, Example 1], it was shown that $\varprojlim \boldsymbol{f}$ is not connected where the graph of f is the union of four straight line intervals, one from $(0,0)$ to $(1/4,1/4)$; one from $(0,0)$ to $(1,0)$; one from $(1,0)$ to $(1,1)$; and one from $(3/4, 1/4)$ to $(1, 1)$. It is worth remarking that, for that function f, although $G(f)$ is connected, $G(f^2)$ is not connected, so Nall's techniques provide an alternative means of proving that $\varprojlim \boldsymbol{f}$ is not connected.

One additional remark is in order before we present our examples. Nall's function is the union of two mappings, one surjective defined on $[0, 1]$ and one defined only on $[1/2, 1]$ (but, of course, not the maps φ and ψ used above to define the example). The reader should contrast Nall's example with [2, Theorem 3.3] from which it follows that an inverse limit on $[0, 1]$ is a continuum when the bonding function is the union of a mapping with a surjective mapping where both maps have domain the entire interval $[0, 1]$.

Before turning to our examples, we present a simple lemma that is useful in our study of composites of upper semi-continuous functions.

Lemma 3.1. *Suppose X is a compact Hausdorff space and $f : X \to 2^X$ is an upper semi-continuous function. If k is a positive integer, then $G(f^{k+1}) = \{(x, y) \in X \times X \mid$ there exists a point $t \in X$ such that $x \in f^{-1}(t)$ and $y \in f^k(t)\}$.*

Proof. $y \in f^{k+1}(x)$ if and only if there is a point $t \in X$ such that $t \in f(x)$ and $y \in f^k(t)$; therefore, we have that $y \in f^{k+1}(x)$ if and only if there is a point $t \in X$ such that $x \in f^{-1}(t)$ and $y \in f^k(t)$. $\square$

In the following example, we specify an upper semi-continuous function from $[0, 1]$ into $2^{[0,1]}$ by identifying its graph. We use $Id_{[0,1]}$ to denote the identity on $[0, 1]$.

Example 3.2. *Suppose n is an integer, $n \geq 2$, and let $f_n : [0, 1] \to 2^{[0,1]}$ be the function whose graph consists of three straight line intervals, one from $(0, 0)$ to $(1, 1)$; one from $(1/n, 0)$ to $(2/n, 2/n)$; and one from $(1/n, 0)$ to $(1, 1 - 1/n)$. Then $G(f_n^k)$ is connected for $1 \leq k < n$ and $G(f_n^n)$ is not connected. (See* Figure 1 *for the graphs of f_2 and f_2^2 and* Figure 2 *for the graphs of f_5 and f_5^5.)*

Proof. Choose a positive integer $n \geq 2$. Observe that f_n is the union of three homeomorphisms:

$g_1 = Id_{[0,1]}$,
$g_2 : [1/n, 1] \to [0, 1 - 1/n]$ where $g_2(x) = x - 1/n$,
$g_3 : [1/n, 2/n] \to [0, 2/n]$ where $g_3(x) = 2x - 2/n$.

It is clear that $G(f_n)$ is connected because $G(g_3)$ intersects both $G(g_1)$ and $G(g_2)$. Note that the points $(0, 0)$ and $(1/n, 0)$ belong to $G(f_n)$, and the entire graph of $G(f_n)$ lies in $[0, 1 - 1/n]^2$ except for two non-separating half-open line intervals lying in the strip $(1 - 1/n, 1] \times [0, 1]$.

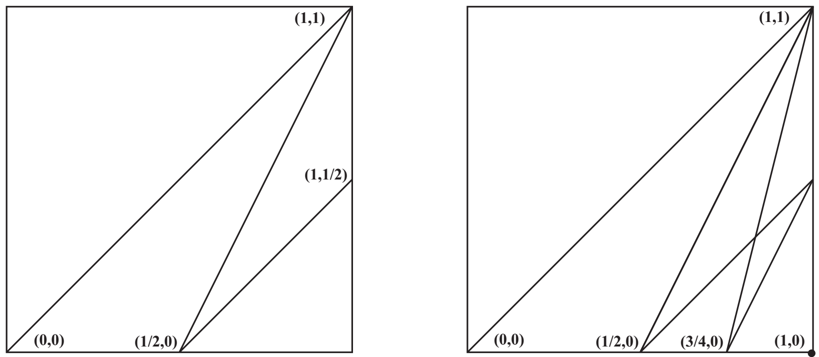

FIGURE 1. *The graphs of* f_2 *and* f_2^2*.*

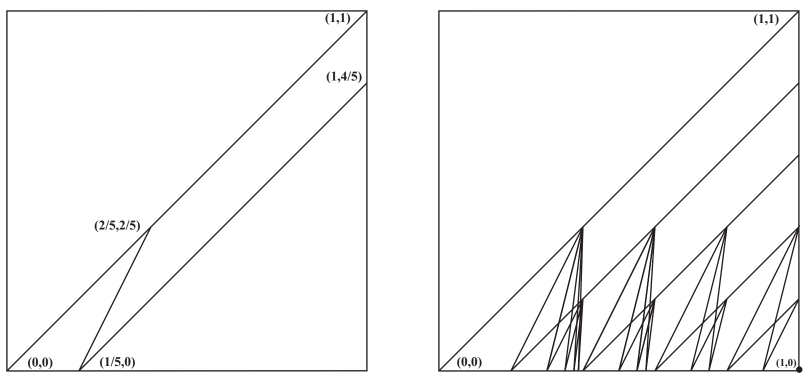

FIGURE 2. *The graphs of* f_n *and* f_n^n *for* $n = 5$*.*

Thus, $G(f_n|[0, 1-1/n])$ is connected. Clearly, $G(f_n|[0, 2/n])$ is connected, whereas $f_n([0, 2/n]) = [0, 2/n]$ and $f_n([0, 1-1/n]) = [0, 1-1/n]$.

Let $\varphi_1 : [0, 1]^2 \to [0, 1]^2$ be given by $\varphi_1(x, y) = (x, y)$, let $\varphi_2 : [0, 1 - 1/n]^2 \to [1/n, 1] \times [0, 1-1/n]$ be given by $\varphi_2(x, y) = (x+1/n, y)$, and let $\varphi_3 : [0, 2/n]^2 \to [1/n, 2/n] \times [0, 2/n]$ be given by $\varphi_3(x, y) = (x/2+1/n, y)$. Note that $p_1 \circ \varphi_1 = g_1^{-1} \circ p_1$, $p_1 \circ \varphi_2 = g_2^{-1} \circ p_1|[0, 1-1/n]^2$, and $p_1 \circ \varphi_3 = g_3^{-1} \circ p_1|[0, 2/n]^2$ where p_1 denotes the projection of $[0, 1]^2$ to its first coordinate space.

We now show that if $1 \leq k \leq n-1$, then $G(f_n^{k+1}) = \varphi_1(G(f_n^k)) \cup \varphi_2(G(f_n^k|[0, 1-1/n]) \cup \varphi_3(G(f_n^k|[0, 2/n]) - \{(2/n, 0)\})$. To see this, first let (x, y) be a point of $G(f_n^{k+1})$. By Lemma 3.1, there is a point $t \in [0, 1]$ such that $x \in f_n^{-1}(t)$ and $y \in f_n^k(t)$. There is an integer i, $1 \leq i \leq 3$, such that $x = g_i^{-1}(t)$ and, for such an i, $(x, y) = \varphi_i(t, y)$ with $(t, y) \in G(f_n^k)$. If $i = 1$, $(x, y) \in \varphi_1(G(f_n^k))$. If $i = 2$, then $0 \le t \le 1 - 1/n$ so $(x, y) \in \varphi_2(G(f_n^k|[0, 1 - 1/n]))$. If $i = 3$ and $(x, y) \ne (2/n, 0)$ then $t \in [0, 2/n]$ and $(x, y) \in \varphi_3(G(f_n^k|[0, 2/n]) - \{(2/n, 0)\})$. In case $(x, y) = (2/n, 0)$, $(x, y) = \varphi_2(1/n, 0)$, so $(x, y) \in \varphi_2(G(f_n^k|[0, 1 - 1/n]))$. On the other hand, if $(x, y) \in \varphi_1(G(f_n^k)) \cup \varphi_2(G(f_n^k|[0, 1-1/n]) \cup \varphi_3(G(f_n^k|[0, 2/n]) - \{(2/n, 0)\})$, then for some i, $1 \le i \le 3$ and some point $t \in [0, 1]$, $x \in g_i^{-1}(t)$ and $y \in f_n^k(t)$. It follows from Lemma 3.1 that $(x, y) \in G(f_n^{k+1})$.

Next, we proceed inductively to show that $G(f_n^k)$ is a connected set containing $(0, 0)$ and $(m/n, 0)$ for $1 \le k \le n - 1$ and $1 \le m \le k$. We have observed this to be true for $k = 1$ because $G(f_n)$ is connected as are $G(f_n|[0, 1 - 1/n])$ and $G(f_n|[0, 2/n])$ and $(0, 0)$ and $(1/n, 0)$ are points of $G(f_n)$.

Suppose j is an integer, $1 \le j < n - 1$, such that $G(f_n^j)$ is a connected set as are $G(f_n^j|[0, 1 - 1/n])$ and $G(f_n^j|[0, 2/n] - \{(2/n, 0)\})$ (we need only to remove the point $(2/n, 0)$ when $j > 1$ because, of course, this point is not in $G(f_n)$). Suppose also that $(0, 0)$ and $(m/n, 0)$ are in $G(f_n^j)$ for $1 \le m \le j$. Then $\varphi_1(G(f_n^j))$ is connected as are $\varphi_2(G(f_n^j|[0, 1 - 1/n]))$ and $\varphi_3(G(f_n^j|[0, 2/n] - \{(2/n, 0)\})$. The point $(1/n, 0)$ belongs to all three of these sets because $\varphi_1(1/n, 0) = (1/n, 0)$ and $(1/n, 0) \in G(f_n^j)$, whereas $\varphi_2(0, 0) = \varphi_3(0, 0) = (1/n, 0)$ and $(0, 0)$ belongs to both $G(f_n^j|[0, 1 - 1/n])$ and $G(f_n^j|[0, 2/n])$. Thus, $G(f_n^{j+1})$ is connected and contains $(0, 0)$ because $\varphi_1(0, 0) = (0, 0)$. Further, the entire graph of f_n^{j+1} lies in $[0, 1-1/n]^2$ except for $j+2$ nonseparating half-open intervals lying in the strip $(1 - 1/n, 1] \times [0, 1]$ (the extra one that is not part of the graph of f_n^j comes from $\varphi_2(G(f_n^j|[0, 1 - 1/n])))$, so $G(f_n^{j+1}|[0, 1 - 1/n])$ is connected. Finally, $G(f_n^{j+1}|[0, 2/n]) - \{(2/n, 0)\}$ is connected. To see this, observe that the portion of $G(f_n^j)$ mapped into $[0, 2/n]^2$ by φ_2 is the union of the straight line interval from $(0, 0)$ to $(1/n, 1/n)$ and the single point $(1/n, 0)$. Thus, $\varphi_2(G(f_n^j|[0, 1 - 1/n])) \cap [0, 2/n]^2$ is the union of the straight line interval from $(1/n, 0)$ to $(2/n, 1/n)$ and the point $(2/n, 0)$. It follows that $G(f_n^{j+1}|[0, 2/n]) - \{(2/n, 0)\}$ is connected being the union of three connected sets $\varphi_1(G(f_n^j|[0, 2/n]) - \{(2/n, 0)\}$, the straight line interval from $(1/n, 0)$ to $(2/n, 1/n)$, and $\varphi_3(G(f_n^j|[0, 2/n] - \{(2/n, 0)\})$ all containing $(1/n, 0)$. Because $(m/n, 0)$ is in $G(f_n^j)$ for $1 \le m \le j$ and $\varphi_2(i/n, 0) = ((i + 1)/n, 0)$ for each i, $1 \le i \le j$, $(m/n, 0) \in G(f_n^{j+1})$ for $1 \le m \le j + 1$.

Therefore, we have that $G(f_n^k)$ is connected for $1 \le k \le n - 1$ and $(1 - 1/n, 0) \in G(f_n^{n-1})$. It now follows that $\varphi_2(1 - 1/n, 0) = (1, 0)$ is in $G(f_n^n)$. However, $(1, 0)$ is an isolated point of $G(f_n^n)$. To see this, observe that $f_n^n(1)$ is a discrete set with minimum 0 and $f_n^{-n}(0)$ is a discrete set with maximum 1. Because $G(f_n^n)$ has an isolated point, it is not connected. $\square$

By way of contrast, there is an upper semi-continuous function $f : [0,1] \to 2^{[0,1]}$ such that $G(f^n)$ is connected for each positive integer n, but $\varprojlim \boldsymbol{f}$ is not connected. One such example, due to Jonathan Meddaugh, is a simple modification to [3, Example 1, p. 126] that attaches the vertical line from $(0,0)$ to $(0,1)$ to that function as shown in Figure 3.

Example 3.3 (Meddaugh)**.** *Let $f : [0,1] \to 2^{[0,1]}$ be the union of five straight line intervals, one from $(0,0)$ to $(0,1)$; one from $(0,0)$ to $(1/4,1/4)$; one from $(0,0)$ to $(1,0)$; one from $(1,0)$ to $(1,1)$; and one from $(3/4,1/4)$ to $(1,1)$. Then $G(f^n)$ is connected for each positive integer n, but G_2 is not connected, so $\varprojlim \boldsymbol{f}$ is not connected.* (See Figure 3.)

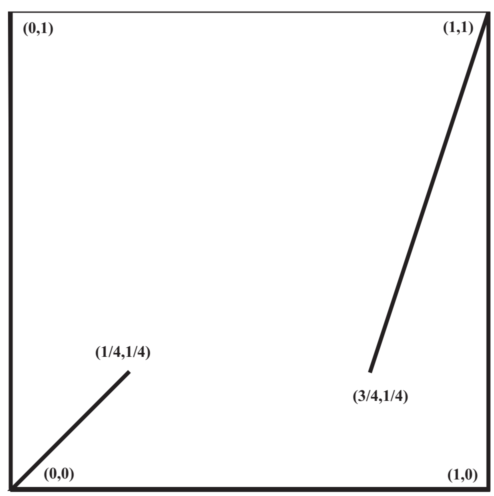

FIGURE 3. *The graph of the function in Meddaugh's example.*

Proof. The graph of f is clearly connected and, for $n > 1$, $G(f^n) = [0,1]^2$, so $G(f^n)$ is connected for each positive integer n. Let N be the set of points $\boldsymbol{x}$ of G_2 such that $x_1 = x_2 = 1/4$ and $x_3 = 3/4$. Then N is a closed subset of G_2. However, $O = (1/8,3/8) \times (1/8,3/8) \times (5/8,7/8) \times [0,1]^\infty$ is an open set such that $N = G_2 \cap O$, so N is also relatively open in G_2. $\square$

Meddaugh's example makes it reasonable to ask about the connectedness of the sets G_i for the functions f_n from Example 3.2. We address this in the following example.

Example 3.4. *Let $n \geq 2$ be an integer and f_n be the function from Example 3.2. Then the set G_k (G'_k, respectively) is connected for $1 \leq k \leq n-1$, but G_n (G'_n, respectively) is not connected.*

Proof. We only deal with the question of the connectedness of $G'_k = \pi_{\{1,2,\ldots,k+1\}}(G_k)$ because G_k is connected if and only if G'_k is connected. It is not difficult to see that, for the function f_2, G'_1 is connected, but $(0, 1/2, 1)$ is an isolated point of G'_2.

Suppose n is an integer greater than 2. Our proof is by induction on the number of composites, but first we make several observations that are useful in the proof.

The function f_n^{-1} is the union of three homeomorphisms:

$h_1 = Id_{[0,1]}$,
$h_2 : [0, 1-1/n] \to [1/n, 1]$ where $h_2(t) = t + 1/n$,
$h_3 : [0, 2/n] \to [1/n, 2/n]$ where $h_3(t) = t/2 + 1/n$.

Observe that for $1 \leq i \leq 3$, $h_i = g_i^{-1}$ where g_1, g_2, and g_3 are the maps defined in Example 3.2.

Suppose j is an integer, $2 \leq j \leq n-1$, and for each $\boldsymbol{x} \in [0,1]^j$, let $\varphi_{j,1} : [0,1]^j \to [0,1]^{j+1}$ be defined by $\varphi_{j,1}(\boldsymbol{x}) = (x_1, \ldots, x_j, h_1(x_j))$, let $\varphi_{j,2} : [0,1]^{j-1} \times [0, 1-1/n] \to [0,1]^{j-1} \times [0, 1-1/n] \times [1/n, 1]$ be defined by $\varphi_{j,2}(\boldsymbol{x}) = (x_1, \ldots, x_j, h_2(x_j))$, and let $\varphi_{j,3} : [0, 2/n]^j \to [0, 2/n]^j \times [1/n, 2/n]$ be defined by $\varphi_{j,3}(\boldsymbol{x}) = (x_1, \ldots, x_j, h_3(x_j))$. Note that if $1 \leq j \leq n-1$ and $1 \leq i \leq 3$, then $\varphi_{j,i}$ is continuous and

$$G'_j = \varphi_{j,1}(G'_{j-1}) \cup \varphi_{j,2}(G'_{j-1} \cap ([0,1]^{j-1} \times [0, 1-1/n])) \cup \varphi_{j,3}(G'_{j-1} \cap [0, 2/n]^j). \qquad (*)$$

Continuing with our observations, for $2 \leq j \leq n-1$, let $K_j = \{\boldsymbol{x} \in G'_{j-1} \mid x_1 = 0 \text{ and } x_j = 1/n\}$ (thus, $\boldsymbol{x} \in K_j$ if and only if there is an integer m, $1 \leq m < j$, such that $x_i = 0$ for $i \leq m$ and $x_i = 1/n$ for $i > m$). Let $D_j = \{\boldsymbol{x} \in [0, 1/n]^j \mid x_i = x_1 \text{ for } 1 \leq i \leq j\}$. So D_j is a subset of the diagonal of $[0,1]^j$. Then $G'_{j-1} \cap [0, 1/n]^j = D_j \cup K_j$. This is easily established by induction because it holds for $j = 2$ and for $2 < m < n-1$; the only point of $G'_m \cap [0, 1/n]^{m+1}$ that is not in $\varphi_{m,1}(G'_{m-1})$ is $\varphi_{m,2}(0, 0, \ldots, 0) = \varphi_{m,3}(0, 0, \ldots, 0) = (0, 0, \ldots, 0, 1/n) \in K_{m+1}$, and $(0, \ldots, 0)$ is the only point of G'_{m-1} mapped by either $\varphi_{j,2}$ or $\varphi_{j,3}$ into $[0, 1/n]^{m+1}$.

Let $L_2 = \emptyset$ and, for $j > 2$, let $L_j = \varphi_{j-1,2}(K_{j-1})$. Then $\boldsymbol{x} \in L_j$ if and only if there is an integer $m, 2 \leq m < j-1$, such that $x_i = 0$ for $1 \leq i \leq m$,

$x_i = 1/n$ for $m < i < j$, and $x_j = 2/n$. Because $h_3(2/n) = 2/n$, we have that $\varphi_{j,3}(L_j) \subseteq \varphi_{j,1}(G'_{j-1})$ from which it follows by using (*) that

$$G'_j = \varphi_{j,1}(G'_{j-1}) \cup \varphi_{j,2}(G'_{j-1} \cap ([0,1]^{j-1} \times [0,1-1/n])) \cup$$
$$\varphi_{j,3}((G'_{j-1} \cap [0,2/n]^j) - L_j). \qquad (**)$$

We now proceed inductively to show that G'_k is connected for $1 \leq k \leq n-1$. Note that G'_1 is a connected set containing $(0,0)$ and $(2/n, 2/n)$, and $G'_1 \cap ([0,1] \times [0,1-1/n])$ and $G'_1 \cap [0,2/n]^2 - L_2$ are both connected (see Figure 4). Although it is easy to see that all three of these are connected sets, in order to give the flavor of one particular part of the argument in our inductive proof, observe that G'_1 contains two points, each having 1 as a second coordinate, namely, $(1-1/n, 1)$ and $(1,1)$. Each of these points is an endpoint of an arc crossing the strip $[0,1] \times [1-1/n, 1]$; the two arcs are mutually exclusive and neither arc separates G'_1. The set $G'_1 \cap ([0,1] \times [0,1-1/n])$ is connected because it is the closure of the complement of the union of these two arcs.

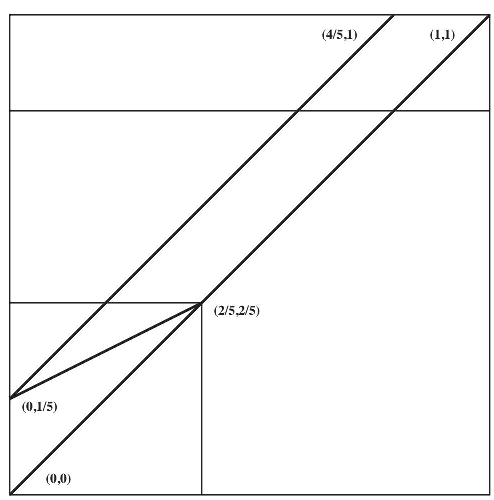

FIGURE 4. *The set $G'_1(\boldsymbol{f_5})$.*

Suppose j is an integer, $2 \leq j \leq n-1$ such that G'_{j-1} is a connected set containing the points $(0,0,\ldots,0)$ and $(2/n, 2/n, \ldots, 2/n)$ of $[0,1]^j$, and $G'_{j-1} \cap ([0,1]^{j-1} \times [0, 1-1/n])$ and $(G'_{j-1} \cap [0, 2/n]^j) - L_j$ are both connected. From (**),

$$G'_j = \varphi_{j,1}(G'_{j-1}) \cup \varphi_{j,2}(G'_{j-1} \cap ([0,1]^{j-1} \times [0, 1-1/n])) \cup \varphi_{j,3}((G'_{j-1} \cap [0, 2/n]^j) - L_j).$$

Each of the three sets in this union is connected, being a continuous image of a connected set. The points $(0, 0, \ldots, 0)$ and $(2/n, 2/n, \ldots, 2/n)$ of $[0,1]^{j+1}$ are elements of $\varphi_{j,1}(G'_{j-1})$ so they are points of G'_j. Because $\varphi_{j,1}(2/n, \ldots, 2/n) = \varphi_{j,3}(2/n, 2/n, \ldots, 2/n) = (2/n, 2/n, \ldots, 2/n)$ (the number of coordinates of these points should be clear from context), the sets $\varphi_{j,1}(G'_{j-1})$ and $\varphi_{j,3}((G'_{j-1} \cap [0, 2/n]^j) - L_j)$ have the point $(2/n, \ldots, 2/n)$ in common. (Each point of L_j has at least one coordinate 0, so $(2/n, \ldots, 2/n) \notin L_j$.) The point $\varphi_{j,2}(0, 0, \ldots, 0) = \varphi_{j,3}(0, 0, \ldots, 0) = (0, 0, \ldots, 0, 1/n)$ is common to $\varphi_{j,2}(G'_{j-1} \cap ([0,1]^{j-1} \times [0, 1-1/n]))$ and $\varphi_{j,3}((G'_{j-1} \cap [0, 2/n]^j) - L_j)$. (The last coordinate of each point of L_j is $2/n$, so $(0, 0, \ldots, 0, 1/n) \notin L_j$.) It follows that G'_j is connected.

To complete the inductive proof, we need to show that if $j < n-1$, then $G'_j \cap [0,1]^j \times [0, 1-1/n]$ and $G'_j \cap [0, 2/n]^{j+1} - L_{j+1}$ are connected.

First, we show that if $j < n-1$, then $G'_j \cap [0,1]^j \times [0, 1-1/n]$ is connected. There are 2^j arcs in the collection $A_{j+1} = \{\alpha \subseteq G'_j \mid \alpha \text{ is an arc and } \pi_{j+1}(\alpha) = [1-1/n, 1]\}$. This is shown by yet another inductive argument that relies on observing that for $1 \le i < n-1$, the set G'_i has 2^{i-1} such arcs arising from $\varphi_{i,1}(G'_{i-1})$ and 2^{i-1} additional ones arising from $\varphi_{i,2}(G'_{i-1} \cap ([0,1]^{i-1} \times [0, 1-1/n]))$. We omit the rest of this detail. Thus, if $\alpha \in A_{j+1}$, then one end point of α has 1 as its last coordinate and $G'_j - \alpha$ is connected. It follows that $G'_j \cap ([0,1]^j \times [0, 1-1/n])$ is connected, being the closure of the complement in G'_j of the union of all of the arcs in A_{j+1}.

Lastly, we show that if $j < n-1$, then $G'_j \cap [0, 2/n]^{j+1} - L_{j+1}$ is connected. This results from the fact that $G'_j \cap [0, 2/n]^{j+1} = \varphi_{j,1}(G'_{j-1} \cap [0, 2/n]^j) \cup \varphi_{j,3}(G'_{j-1} \cap [0, 2/n]^j) \cup \varphi_{j,2}(G'_{j-1} \cap [0, 1/n]^j)$. Because $\varphi_{j,1}(G'_{j-1} \cap [0, 2/n]^j)$ and $\varphi_{j,3}(G'_{j-1} \cap [0, 2/n]^j)$ are connected with $(2/n, 2/n, \ldots, 2/n)$ in common, their union is connected. The set $\varphi_{j,2}(G'_{j-1} \cap [0, 1/n]^j) = \varphi_{j,2}(D_j \cup K_j) = \varphi_{j,2}(D_j) \cup L_{j+1}$. The set $\varphi_{j,2}(D_j)$ is a connected subset of $G'_j \cap [0, 2/n]^{j+1}$ containing the point $(0, 0, \ldots, 0, 1/n)$ in common with $\varphi_{j,3}(G'_{j-1} \cap [0, 2/n]^j)$.

This completes the inductive proof, and the connectedness of G'_k for $1 \le k \le n-1$ is now established. The set G'_n is not connected because it contains $(0, 1/n, 2/n, \ldots, 1)$ as an isolated point. $\square$

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