INVERSE LIMIT REPRESENTATIONS OF NON-PLANAR TREE-LIKE CONTINUA

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ABSTRACT. In this paper we present a collection of non-planar tree-like continua with the property that if n is a positive integer there is a continuum in the collection such that the first n stages of a construction of the continuum can be carried out in the plane. Each continuum in the collection is obtained as an inverse limit on a single planar tree with a single bonding map.

1. INTRODUCTION

Drawing pictures of bonding maps in inverse limit sequences when the factor spaces are not arcs presents a challenge. Even if all of the factor spaces are as elementary as a simple triod, the Cartesian product containing the graph of the bonding map does not embed in Euclidean three dimesional space. Fortunately, one can obtain a schematic picture of the bonding map that gives information not only about the mapping but also about the inverse limit. Over the years, the author has made extensive use of a simple scheme to assist in analyzing the resulting inverse limit. One advantage of this scheme is that it is closely tied to chaining (linear, tree, circular, etc.) of the inverse limit and the patterns the chains follow one in the other as well as the construction of continua in Euclidean spaces homeomorphic to the inverse limit. We carefully describe this scheme in section 2.

In this paper we present a collection of non-planar tree-like continua each of which is obtained as an inverse limit on a single planar tree with a single bonding map. The collection has the property that if n is a positive integer there is a continuum in the collection such that the first n stages of a construction of a homeomorphic copy of the continuum can be carried out in the plane. The "cause" of this phenomenon appears to lie in the different embeddings of the domain tree and the "fattened" range in our schematics. Informally, we describe this phenomenon as being able to draw the first n stages of a construction of the continuum in the plane even though the continuum cannot be embedded in the plane. First we give some basic definitions.

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By a *continuum* we mean a compact, connected subset of a metric space. If $X_1, X_2, X_3, \ldots$ is a sequence of continua and $f_1, f_2, f_3, \ldots$ is a sequence of mappings such that $f_i : X_{i+1} \to X_i$, the *inverse limit* of the inverse sequence $\{X_i, f_i\}$ is the subset of the product $\prod_{i>0} X_i$ containing the point $(x_1, x_2, x_3, \ldots)$ if and only if $f_i(x_{i+1}) = x_i$. The inverse limit of this inverse limit sequence is denoted $\varprojlim\{X_i, f_i\}$. The continua X_i will be called *factor spaces* and the mappings f_i *bonding maps* of the inverse limit sequence. If X is a continuum and $f : X \to X$ is a mapping, by $\varprojlim\{X, f\}$ we mean the inverse limit of the inverse sequence $\{X_i, f_i\}$ where, for each i, $X_i = X$ and $f_i = f$. We denote by π_i the projection of the inverse limit into the ith factor space X_i. A *ray* is a topological image of $[0, 1)$. If M is a continuum that is the union of a ray R and a continuum K such that $M = \overline{R}$ and $\overline{R} - R = K$, we call K the *remainder* of R in M. We use the notation $f : X \twoheadrightarrow Y$ to denote that f is a mapping of X onto Y.

2. An Example

In this section we describe a plane continuum consisting of a ray with a simple triod as remainder and give two inverse limit representations of this continuum (see Figure 1). This construction is carried out in rather full detail for the benefit of the reader who is not familiar with such constructions. A more experienced reader will likely take note of our geometric description of the continuum and its two inverse limit representations and move on to the final section of the article. The continuum that we shall denote by M is constructed in the plane as follows. Let H denote the simple triod that is the union of three arcs J_1, J_2 and J_3 lying on the coordinate axes each having the origin as one end point. The other end point of J_1 is the point $(1, 0)$, the other end point of J_2 is $(0, 1)$ and the other end point of J_3 is $(-1, 0)$. Denote by P_0 the point $(0, -2)$ and by P_i the point $(0, -1/i)$ for $i = 1, 2, 3, \ldots$. Further, let $Q_i = (1 + 1/i, 0)$, $R_i = (1/i, 1/i)$, $S_i = (0, 1 + 1/i)$, $T_i = (-1/i, 1/i)$ and $U_i = (-1 - 1/i, 0)$ for $i = 1, 2, 3, \ldots$. If P and Q are points in the plane, we denote by PQ the straight line interval joining P and Q. Let $I_0, I_1, I_2, \ldots$ be a sequence of straight line intervals such that $I_0 = P_0P_1$ and, for $j \geq 0$, let $I_{6j+1} = P_{j+1}Q_{j+1}$, $I_{6j+2} = Q_{j+1}R_{j+1}$, $I_{6j+3} = R_{j+1}S_{j+1}$, $I_{6j+4} = S_{j+1}T_{j+1}$, $I_{6j+5} = T_{j+1}U_{j+1}$ and $I_{6j+6} = U_{j+1}P_{j+2}$. Then, M is the union of J_1, J_2 and J_3 and the ray that is the union of $I_0, I_1, I_2, \ldots$.

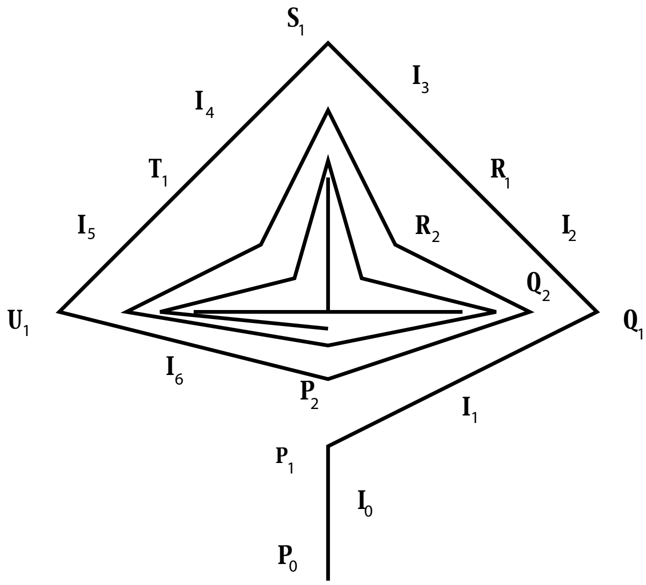

FIGURE 1.

We now describe an inverse limit sequence whose inverse limit is M. Let X denote a simple 4-od in the plane that is the union of four unit intervals intersecting at the origin and lying on the coordinate axes. We shall use the following labeling of X: denote the origin by O and let $A = (1, 0)$, $B = (0, 1)$, $C = (-1, 0)$ and $D = (0, -1)$. Then $X = OA \cup OB \cup OC \cup OD$. If $P = (x, y)$ is a point in the plane and t is a real number, we denote the point (tx, ty) by tP. Let $f : X \twoheadrightarrow X$ be given by $f(tP) = tP$ for $0 \le t \le 1$ and P in $\{A, B, C\}$ while

$$f(tD) = \begin{cases} 7tC & \text{if } 0 \le t \le 1/7 \\ (-7t+2)C & \text{if } 1/7 \le t \le 2/7 \\ (7t-2)B & \text{if } 2/7 \le t \le 3/7 \\ (-7t+4)B & \text{if } 3/7 \le t \le 4/7 \\ (7t-4)A & \text{if } 4/7 \le t \le 5/7 \\ (-7t+6)A & \text{if } 5/7 \le t \le 6/7 \\ (7t-6)D & \text{if } 6/7 \le t \le 1 \end{cases}$$

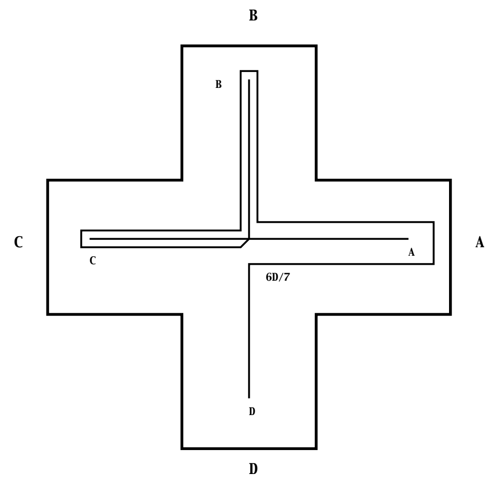

FIGURE 2.

Our schematic representation of f is shown in Figure 2. In the schematic, the range 4-od is drawn "fattened" into a disk with four arms. The domain 4-od is drawn "folded" inside the disk. Since f is the identity on $OA \cup OB \cup OC$ this triod is simply drawn inside the corresponding arms of the disk. The arm OD is thrown over all of X in the following way: first break OD into seven subintervals of equal length, then map the first two subintervals out OC and back, the second two out OB and back, the third two out OA and back and the seventh subinterval out OD. That $(6/7)D$ is mapped to O is indicated by the $6D/7$ mark on the folded 4-od. Marking the other points of the subdivision of OD in the picture would only serve to clutter the picture, but, for example, $5D/7$ could be marked along the vertical segment OD of the folded 4-od lying in the A-arm of the disk. The short vertical and horizontal intervals are present in the schematic only for minor technical convenience in the drawing and may be more or less ignored and could have been drawn as sharp reversals had we so wished.

It is not difficult to see that $\varprojlim\{X, f\}$ is homeomorphic to the continuum M of Figure 1. One way to see this would be to use a chaining construction to build a homeomorphism between the continua. However, we shall indicate another means to construct a homeomorphism. Let α_0 be the inverse limit of the inverse sequence with factors $OD, (6D/7)D, \ldots$ and bonding maps restrictions of f to the designated factor space. Since the second factor lies in OD all subsequent factors are automatically determined and lie in OD. In each of the inverse sequences described below, once a factor is chosen to lie in OD, the subsequent factors are similarly all determined for the same reason. As well, it should be understood that we restrict the bonding map f to the appropriate factor space in each of these inverse sequences. Suppose j is a non-negative integer. Let α_{6j+1} be the arc that is the inverse limit of the sequence having OA as its first $j+1$ factors and then $(5D/7)(6D/7)$ as its next factor, ... Let α_{6j+2} be the inverse limit of the sequence having OA as its first $j+1$ factors and then $(4D/7)(5D/7)$ as its next factor, ... Let α_{6j+3} be the inverse limit of the sequence having OB as its first $j+1$ factors and then $(3D/7)(4D/7), \ldots$. Let α_{6j+4} be the inverse limit of the sequence having OB as its first $j+1$ factors and then $(2D/7)(3D/7), \ldots$. Let α_{6j+5} be the inverse limit of the sequence having OC as its first $j+1$ factors and then $(D/7)(2D/7), \ldots$. Finally, let α_{6j+6} be the inverse limit of the sequence having OC as its first $j+1$ factors and then $O(D/7), \ldots$. Let L_1 be the inverse limit of the sequence having all of its factors OA, L_2 the inverse limit of the sequence having all of its factors OB and L_3 the inverse limit of the sequence having all of its factors OC. By mapping L_i linearly onto J_i for $i = 1, 2, 3$ and α_i linearly onto I_i for $i \geq 0$ (being careful to match the order on the arcs α_i and I_i in this process) we obtain a homeomorphism of $\varprojlim\{X, f\}$ onto M.

A different representation of the continuum M may be obtained using another mapping of the 4-od. Define $g : X \twoheadrightarrow X$ by $g(tA) = tB$, $g(tB) = tC$ and $g(tC) = tA$ for $0 \leq t \leq 1$ while

$$g(tD) = \begin{cases} 3tA & \text{if } 0 \leq t \leq 1/3 \\ (-3t+2)A & \text{if } 1/3 \leq t \leq 2/3 \\ (3t-2)D & \text{if } 2/3 \leq t \leq 1 \end{cases}$$

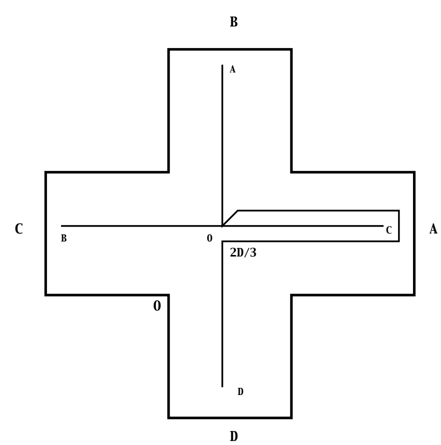

FIGURE 3.

A schematic representation of g is shown in Figure 3. As before the range 4-od is drawn "fattened" into a disk with four arms and the domain 4-od is drawn "folded" inside this disk. The rotation on the triod $OA \cup OB \cup OC$ is represented by its labeling inside the disk with four arms. The leg OD of X is broken into thirds with the first two thrown over OC and back while the final third is thrown over OD. That $(2/3)D$ is mapped to O is indicated by the $2D/3$ placed at the final turn in the D-leg of the domain 4-od.

One advantage to our schematic system arises in drawing compositions. To see $g^2 = g \circ g$, we "fatten" the the folded 4-od into a disk with four arms (drawn inside the original "fattened" disk) and then draw the domain 4-od of g^2 inside this disk. By subsequently erasing the intermediate disk, one arrives at a picture of g^2. The two steps of this process are shown in Figures 4 and 5.

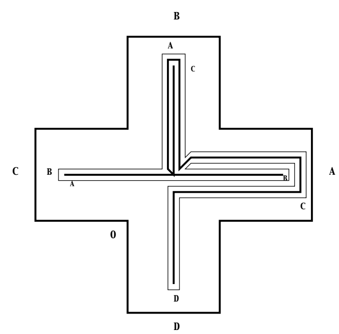

FIGURE 4.

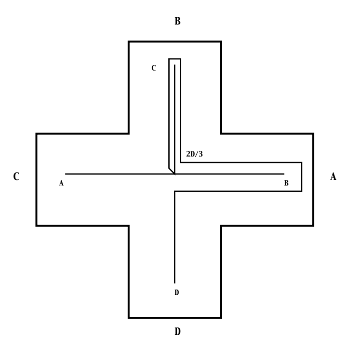

FIGURE 5.

Finally, in a similar manner we obtain a schematic of g^3 (see Figure 6). One cannot help noticing the similarity between the schematic for g^3 and that for f. Indeed, the two inverse limits $\varprojlim\{X, f^3\}$ and $\varprojlim\{X, g\}$ are homeomorphic. One way to see this is as follows. Denote by $h : [0, 1] \twoheadrightarrow [0, 1]$ the piecewise linear homeomorphism whose graph consists of three straight line intervals joining $(0, 0)$ and $(2/7, 2/3)$, $(2/7, 2/3)$ and $(4/7, 8/9)$, and $(4/7, 8/9)$ and $(1, 1)$, respectively. Let $h_1 = h$ and for $n \geq 1$ let

$$h_{n+1}(t) = \begin{cases} h_n(t) & \text{if } 0 \leq t \leq 6/7 \\ (1/27)h_n(7t-6) + 26/27 & \text{if } 6/7 \leq t \leq 1 \end{cases}$$

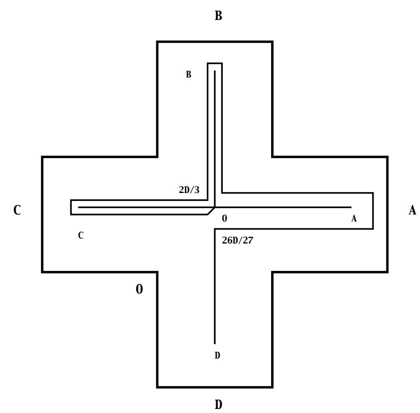

FIGURE 6.

Let $\phi_1 : X \to X$ be the identity on X and, for $i \geq 2$, let $\phi_i : X \twoheadrightarrow X$ be given by

$$\phi_i(tP) = \begin{cases} tP & \text{for } P \text{ in } \{A, B, C\} \\ h_{i-1}(t)D & \text{for } P = D \end{cases}$$

where $0 \leq t \leq 1$. Then, $\phi_n \circ f = g^3 \circ \phi_{n+1}$ for $n = 1, 2, 3, \ldots$ so the sequence of homeomorphisms $\phi_1, \phi_2, \phi_3, \ldots$ induces a homeomorphism from $\varprojlim\{X, f\}$ onto $\varprojlim\{X, g^3\}$, [3, Theorem 5.2]. Since by the subsequence theorem [2, Corollary 1.7.1, p. 11] $\varprojlim\{X, g^3\}$ and $\varprojlim\{X, g\}$ are homeomorphic we have that $\varprojlim\{X, f\}$ and $\varprojlim\{X, g\}$ are homeomorphic.

3. THE COLLECTION OF CONTINUA

R H Bing observed that if an arc having O as an endpoint is attached to the continuum M (shown in Figure 1) so that O is the only point common to this arc and M, the resulting continuum cannot be embedded in the plane [1, Example 1, pp. 654–55]. Such a continuum may be constructed as an inverse limit in the following way:

Let E_1, E_2, E_3, E_4 and E_5 be five evenly spaced points on the unit circle in the plane numbered counter-clockwise with $E_1 = (1, 0)$. Let X denote the 5-od, $OE_1 \cup OE_2 \cup OE_3, \cup OE_4 \cup OE_5$. Let f be the map of X onto itself that simply rotates OE_1 onto OE_2, OE_2 onto OE_3, and OE_3 onto OE_1, while f is the identity on OE_4. To define f on OE_5 we subdivide this interval into thirds. Then, f maps OE_5 linearly out OE_1 and back on the first two subintervals and then out OE_5 on the final one. A schematic representation of f is shown in Figure 7. Note that it is easy to draw schematics of f, f^2 and f^3 in the plane (a schematic for f^3 is shown in Figure 8) even though $\varprojlim\{X, f\}$ is a non-planar continuum.

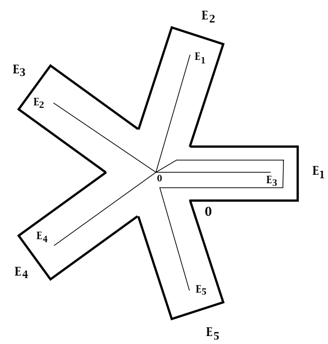

FIGURE 7.

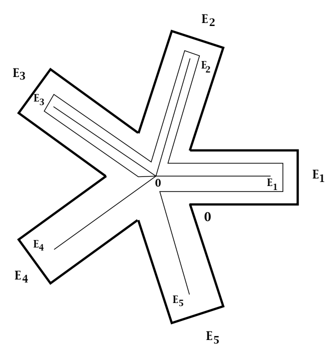

FIGURE 8.

We now describe the collection of examples. Each example is a simple adaptation of the example we just described. Let n be a positive integer, $n \geq 3$. Let $E_1, E_2, \ldots, E_{n+2}$ be $n+2$ evenly spaced points on the unit circle in the plane numbered counter-clockwise with $E_1 = (1, 0)$. Let X be the $(n+2)$-od, $OE_1 \cup OE_2 \cup \ldots OE_{n+2}$. Let f be the map of X onto itself that simply rotates OE_i onto OE_{i+1} for $1 \leq i \leq n-1$ and OE_n onto OE_1, while f is the identity on OE_{n+1}. To define f on OE_{n+2} we subdivide this interval into thirds. Then, f maps OE_{n+2} linearly out OE_1 and back on the first two subintervals and then out OE_{n+2} on the final one. Then $\varprojlim\{X, f\}$ is a non-planar continuum that is the union of an arc $\alpha = \varprojlim\{OE_{n+1}, f|OE_{n+1}$ and a ray with remainder the n-od that results from the inverse limit on the n-od $OE_1 \cup OE_2 \cup \ldots \cup OE_n$ using the restriction of f. The arc α and the continuum that is the union of the ray and this n-od intersect only at $(O, O, O, \ldots)$. It is possible to draw schematics of $f, f^2, \ldots, f^n$ in the plane even though $\varprojlim\{X, f\}$ is a non-planar continuum.

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