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# INVERSE LIMITS WITH UPPER SEMI-CONTINUOUS BONDING FUNCTIONS: PROBLEMS AND SOME PARTIAL SOLUTIONS

W. T. INGRAM

Abstract. By means of numerous examples we call attention to several problems in the theory of inverse limits with upper semi-continuous bonding functions. Along with the problems we present a few partial solutions. Most of the problems we discuss arise from the failure of certain theorems from the theory of inverse limits with mappings to carry over to the setting of inverse limits with set-valued functions.

## 1. Introduction

Many of the tools employed by researchers in inverse limits with mappings simply do not carry over to inverse limits with upper semi-continuous bonding functions. Although this can be frustrating to an experienced researcher in ordinary inverse limits, it presents a golden opportunity for research. In this article we shall explore a number of these areas by surveying some of these differences and posing several questions. For some of the questions, we present partial solutions, but our main emphasis will be on examples illustrating where the tools fail and on posing questions for further exploration. In general there are two kinds of solutions we think would be of interest. One would be a solution in terms of the bonding functions giving the potential users of the results the most obvious access to their use. The other sort of solution is one for

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which the corresponding result for inverse limits with mappings is a corollary. Ideally, a solution would satisfy both of these criteria.

If $Y$ is a compact Hausdorff space, we denote the collection of closed subsets of $Y$ by $2^Y$ and the collection of closed and connected subsets of $Y$ by $C(Y)$. If each of $X$ and $Y$ is a compact Hausdorff space, a function $f : X \to 2^Y$ is said to be *upper semi-continuous at the point $x$ of $X$* provided if $V$ is an open set in $Y$ that contains $f(x)$, then there is an open set $U \subseteq X$ containing $x$ such that if $t$ is a point of $U$, then $f(t) \subseteq V$. A function $f : X \to 2^Y$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of $X$. If $f : X \to 2^Y$ is a set-valued function, we shall say that $f$ is *surjective* provided for each $y \in Y$ there is a point $x \in X$ such that $y \in f(x)$. If $f : X \to 2^Y$ is a set-valued function, by the *graph* of $f$, denoted $G(f)$, we mean the subset of $X \times Y$ that contains the point $(x, y)$ if and only if $y \in f(x)$. It is known that if $M$ is a subset of $X \times Y$ such that $X$ is the projection of $M$ to its set of first coordinates, then $M$ is closed if and only if $M$ is the graph of an upper semi-continuous function, [4, Theorem 2.1].

Suppose $X_1, X_2, X_3, \ldots$ is a sequence of compact Hausdorff spaces and $f_1, f_2, f_3, \ldots$ is a sequence of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ for each positive integer $i$. By the inverse limit of the sequence $f_1, f_2, f_3, \ldots$, denoted $\varprojlim \boldsymbol{f}$, we mean the subset of $\prod_{i>0} X_i$ that contains the point $(x_1, x_2, x_3, \ldots)$ if and only if $x_i \in f_i(x_{i+1})$ for each positive integer $i$. (Throughout this paper we shall denote sequences with boldface type and the terms of sequences in italic type.) The pair of sequences $\{X_i, f_i\}$ is called an inverse limit sequence, the spaces $X_i$ factor spaces and the functions $f_i$ bonding functions. If each $X_i$ is the compact Hausdorff space $X$ and each $f_i$ is the upper semi-continuous function $f : X \to 2^X$ (i. e., we have an inverse limit sequence with a single bonding function), we still denote the inverse limit by $\varprojlim \boldsymbol{f}$. We shall denote the projection from the inverse limit into the $i^{th}$ factor space by $\pi_i$. Inverse limits of compact Hausdorff spaces with upper semi-continuous bonding functions are non-empty and compact, [4, Theorem 3.2]. We shall call a compact, connected Hausdorff space a *Hausdorff continuum* and we shall use the term *continuum* to mean a compact, connected metric space. If $f : X \to C(Y)$ is upper semi-continuous, we say that $f$ is Hausdorff continuum-valued;

if $Y = [0, 1]$, we say $f$ is interval-valued. In the examples involving inverse limits of set-valued functions on $[0, 1]$, we shall use $\mathcal{Q}$ to denote $[0, 1]^\infty = [0, 1] \times [0, 1] \times [0, 1] \times \dots$.

## 2. Connected inverse limits

Armed with the information that the inverse limit of upper semi-continuous functions on compact Hausdorff spaces is compact, one of the first questions someone in continuum theory is likely to ask about inverse limits with set-valued bonding functions is whether the inverse limit is connected if the factor spaces are connected. This turns out not to be the case even if all of the factor spaces are the interval $[0, 1]$ and there is a single bonding function having a connected graph, see [4, Example 1, p. 126]. We include this example without proof. However, later we shall provide another such example, see Example 2.8, for which our proof that it is not connected is virtually identical to the proof given in [4] that Example 2.1 is not connected.

**Example 2.1.** Let $f : [0, 1] \to 2^{[0,1]}$ be the function whose graph consists of four straight line intervals, one from (1/4,1/4) to (0,0), one from (0,0) to (1,0), one from (1,0) to (1,1), and one from (1,1) to (3/4,1/4). The graph of $f$ is connected and $\varprojlim \boldsymbol{f}$ is not connected. (See Figure 1 for the graph of $f$.)

Sufficient conditions that inverse limits of upper semi-continuous functions on Hausdorff spaces be connected are given in Theorem 4.7 and Theorem 4.8 of [4]. Recent conversations with Rob Roe reminded the author that we are far from understanding when inverse limits with upper semi-continuous bonding functions are connected. For functions not satisfying the conditions of the aforementioned theorems, ad hoc arguments that the inverse limit is connected have been supplied. In a recent article in Topology Proceedings, the author did just that showing that under certain conditions a single upper semi-continuous function on a continuum such that the function is a union of mappings produces a connected inverse limit, [3, Theorem 3.3 and Theorem 4.2]. Examples 2.6 and 2.7 given below also require arguments tailored to the examples since the graphs of the bonding functions do not fit the conditions of the theorems from [4].

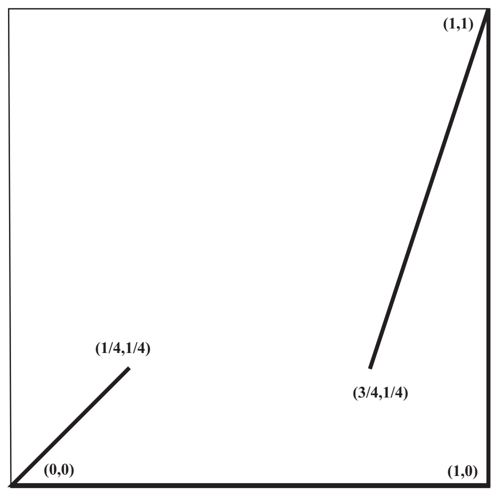


FIGURE 1. *Graph of the bonding function (Example 2.1)*

**Problem 2.2.** *Suppose* $\boldsymbol{f}$ *is a sequence of upper semi-continuous functions on Hausdorff continua. Find necessary and sufficient conditions (preferably on the bonding functions) such that* $\varprojlim \boldsymbol{f}$ *is connected.*

Problem 2.2 is probably far too general a question to be solved completely, perhaps for a long time. Asking for such a result in the metric setting may be as well. Perhaps more tractable is the following problem for which an answer, even in the case that $X = [0,1]$, would be of considerable interest.

**Problem 2.3.** *Suppose* $X$ *is a compact metric space and* $f : X \to 2^X$ *is an upper semi-continuous function. Find sufficient conditions on* $f$ *such that* $\varprojlim \boldsymbol{f}$ *is connected.*

Viewing inverse limits with upper semi-continuous bonding functions as inverse limits with closed subsets of product spaces goes back to William S. Mahavier's original article, [6]. At the meeting of the American Mathematical Society at Baylor in October 2009, Van Nall announced some sufficient conditions on closed subsets of product spaces yielding that the inverse limit is connected.

As mentioned above, in [3] the author showed that an inverse limit of an upper semi-continuous function on $[0,1]$ such that the function is the union of mappings one of which is surjective is connected. Recently, we observed that the function from Example 2.1 is the union of two upper semi-continuous functions such that each of them has a connected inverse limit. Specifically, it is the union of the functions from Example 2.6 and Example 2.7. This shows that the result from [3] does not generalize to the inverse limit of an upper semi-continuous function that is the union of surjective upper semi-continuous functions on $[0,1]$ each having a connected inverse limit. In the proofs that the functions in Example 2.6 and Example 2.7 produce continua, we make use of the following observation. Its proof is left to the reader.

**Theorem 2.4.** *Suppose $X$ is a compact Hausdorff space and $f : X \to 2^X$ is an upper semi-continuous function. If $Y$ is a closed subset of $X$ and $g : Y \to 2^Y$ is an upper semi-continuous function such that $G(g) \subseteq G(f)$, then $\varprojlim \boldsymbol{g}$ is a closed subset of $\varprojlim \boldsymbol{f}$.*

Although it is not difficult to verify that the inverse limit of the function in the following example is an arc, we include a proof since we make use of this arc in some of the examples that follow.

**Example 2.5.** Let $f : [0,1] \to C([0,1])$ be the function whose graph is the union of two straight line intervals one from (0,0) to (1,0) and the other from (1,0) to (1,1). Then $\varprojlim \boldsymbol{f}$ is an arc. (See Figure 2 for the graph of $f$. This inverse limit is a subset of the continuum depicted in Figure 4.)

*Proof:* Let $A = \varprojlim \boldsymbol{f}$ and let $A_0 = \{\boldsymbol{x} \in A \mid x_k = 1$ for $k > 1\}$ and $\boldsymbol{p_0} = (1,1,1,\dots)$. For each positive integer $n$, let $A_n = \{\boldsymbol{x} \in A \mid x_k = 0$ for $1 \le k \le n$ and $x_k = 1$ for $k > n+1\}$ and denote by $\boldsymbol{p_n}$ the point of $A$ such that $\pi_j(\boldsymbol{p_n}) = 0$ for $1 \le j \le n$ and $\pi_j(\boldsymbol{p_n}) = 1$ for $j > n$. Observe that, for each integer $i \ge 0$, $A_i$ is an arc and $A_i \cap A_{i+1} = \{\boldsymbol{p_{i+1}}\}$. Moreover, $A = (\bigcup_{i\ge 0} A_i) \cup \{(0,0,0,\dots)\}$. Since each point of $A$ other than $(0,0,0,\dots)$ and $(1,1,1,\dots)$ separates $A$, $A$ is an arc. □

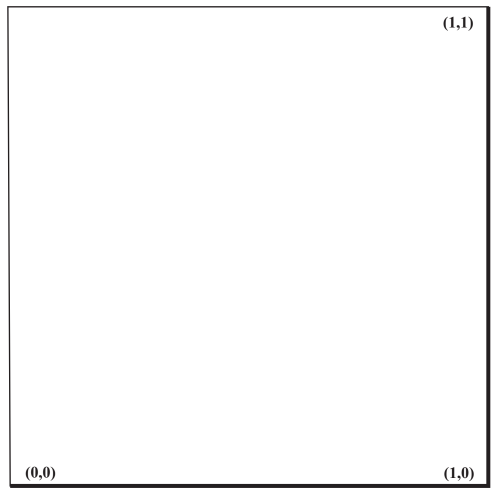


FIGURE 2. *Graph of the bonding function (Example 2.5)*

**Example 2.6.** Let $f_1 : [0,1] \to 2^{[0,1]}$ be the function whose graph is the union of three straight line intervals, one from (1/4,1/4) to (0,0), one from (0,0) to (1,0), and one from (1,0) to (1,1). Then $\varprojlim \boldsymbol{f_1}$ is connected. (See Figure 3 for the graph of $f_1$ and Figure 4 for a picture of this inverse limit.)

*Proof:* Let $M = \varprojlim \boldsymbol{f_1}$. Let $f$ be the function from Example 2.5 and $A$ be the arc that is its inverse limit. By Theorem 2.4, $A \subseteq M$. Let $i$ and $j$ be integers with $i \geq 2$ and $0 \leq j < i-1$. Let $C_{i\,j} = \{\boldsymbol{x} \in M \mid x_i \in [0,1/4], x_k = x_i \text{ for } j < k \leq i, x_k = 1 \text{ for } k > i \text{ and if } j > 0, x_k = 0 \text{ for } 1 \leq k \leq j\}$. Let $B_0 = \{\boldsymbol{x} \in M \mid x_k \in [0,1/4]$ and $x_{k+1} = x_k$ for each positive integer $k\}$ and, for each positive integer $i$, let $B_i = \{\boldsymbol{x} \in M \mid x_{i+1} \in [0,1/4]$ and $x_k = x_{i+1}$ for $k \geq i+1$ and $x_k = 0$ for $k \leq i\}$. Note that $F = \bigcup_{i \geq 0} B_i$ is a fan intersecting $A$ at $(0,0,0,\dots)$. Further, if $i$ and $j$ are integers with $i \geq 2$ and $0 \leq j < i-1$, then $C_{i\,j}$ intersects $A$ at the point $\boldsymbol{p_i}$ where the first $i$ coordinates of $\boldsymbol{p_i}$ are 0 and the remaining coordinates are 1. To see that $M$ is connected, one need only observe that if $\boldsymbol{x} \in M - (A \cup F)$, then $\boldsymbol{x}$ is in $C_{i\,j}$ for some $i$ and $j$. $\square$

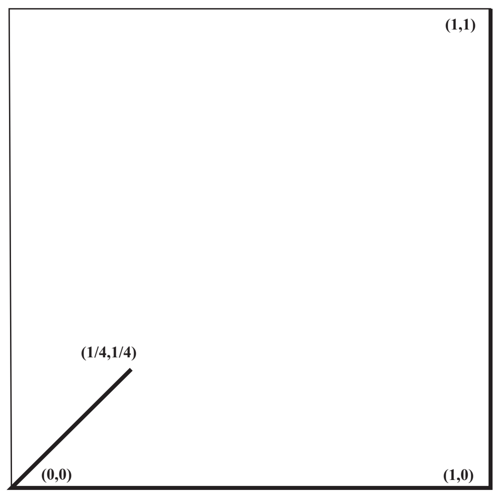


FIGURE 3. *Graph of the bonding function (Example 2.6)*

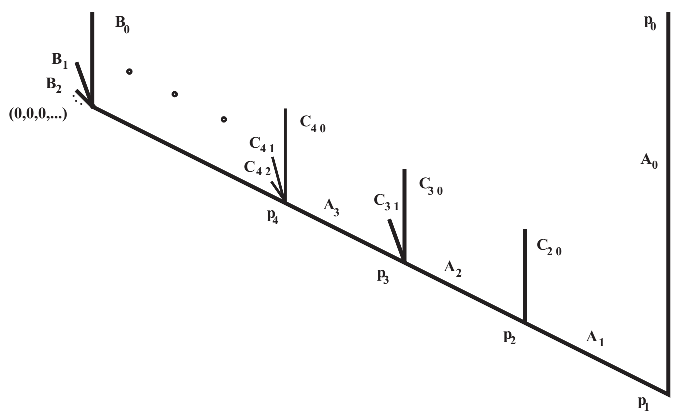


FIGURE 4. *Model for the inverse limit (Example 2.6)*

**Example 2.7.** Let $f_2 : [0,1] \to 2^{[0,1]}$ be the function whose graph is the union of three straight line intervals, one from (0,0) to (1,0), one from (1,0) to (1,1), and one from (1,1) to (3/4,1/4). Then $\varprojlim \boldsymbol{f_2}$ is connected. (See Figure 5 for a graph of $f_2$ and Figure 6 for a picture of this inverse limit.)

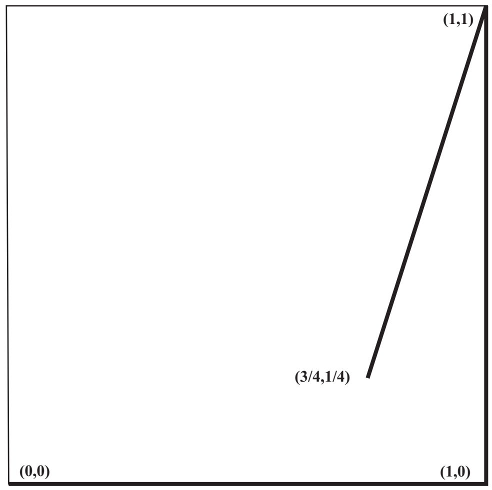


FIGURE 5. *Graph of the function (Example 2.7)*

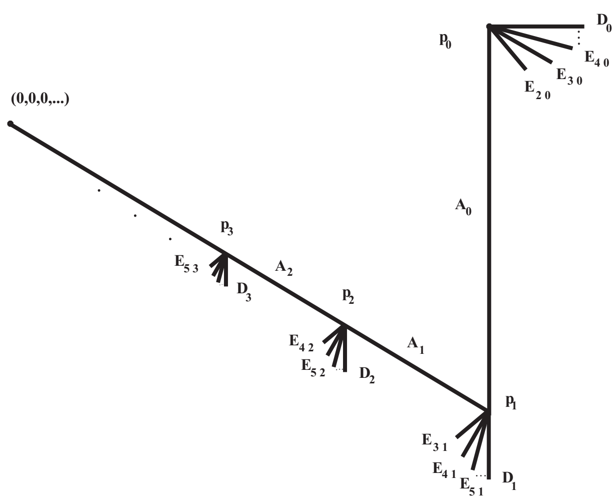


FIGURE 6. *Model for the inverse limit (Example 2.7)*

*Proof:* Let $M = \varprojlim \boldsymbol{f_2}$. Let $f$ be the function from Example 2.5 and $A$ be the arc that is its inverse limit. By Theorem 2.4, $A \subseteq M$. Let $\boldsymbol{p_0}$ be the point $(1, 1, 1, \dots)$ and, for each positive integer $j$, let $\boldsymbol{p_j}$ be the point of $M$ whose first $j$ coordinates are 0 and all other coordinates are 1. Each point of the sequence $\boldsymbol{p_0, p_1, p_2}, \ldots$ is a point of $A$. For $j \geq 0$, let $D_j = \{\boldsymbol{x} \in M \mid 1/4 \leq x_{j+1} \leq 1, x_{k+1} = (x_k + 2)/3$ for $k > j$ and, if $j > 0, x_k = 0$ for $1 \leq k \leq j\}$. For each integer $j \geq 0$ and each integer $i$ such that $i \geq j + 2$, let $E_{i\,j} = \{\boldsymbol{x} \in M \mid 1/4 \leq x_{j+1} \leq 1, x_{k+1} = (x_k + 2)/3$ for $j + 1 \leq k < i, x_k = 1$ for $k > i$ and, if $j > 0, x_k = 0$ for $1 \leq k \leq j\}$. For each $n \geq 0$, $F_n = D_n \cup (\bigcup_{k>n+1} E_{k\,n})$ is a fan with vertex $\boldsymbol{p_n}$. Note that $M = A \cup (\bigcup_{n \geq 0} F_n)$ so $M$ is connected. □

Recently, the author considered the question whether an upper semi-continuous function that is the union of a mapping and an upper semi-continuous function that has a connected inverse limit produces a connected inverse limit. We next provide an example showing that even in this case the inverse limit need not be connected. The proof is essentially identical to the proof of Example 1 in [4], but we include it for the sake of completeness.

**Example 2.8.** Let $f_1 : [0,1] \to 2^{[0,1]}$ be the function whose graph is the union of three straight line intervals, one from (1/4,1/4) to (0,0), one from (0,0) to (1,0), and one from (1,0) to (1,1). (Note that $f_1$ is the function from Example 2.6.) Let $g : [0,1] \to [0,1]$ be the mapping whose graph is the union of three straight line intervals, one from (0,1) to (3/4,1/4), one from (3/4,1/4) to (7/8,1/2), and one from (7/8,1/2) to (1,0). Let $f : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function whose graph is the union of the graphs of $f_1$ and $g$. Then $\varprojlim \boldsymbol{f}$ is not connected. (See Figure 7 for the graph of the bonding function.)

*Proof:* Let $N$ be the set of all points $\boldsymbol{p}$ of $\varprojlim \boldsymbol{f}$ such that $p_1 = p_2 = 1/4$ and $p_3 = 3/4$. Note that $N$ is closed. Let $R = R_1 \times R_2 \times R_3 \times \mathcal{Q}$ be the open set where $R_1 = R_2 = (1/8, 3/8)$ and $R_3 = (5/8, 7/8)$ and observe that $N$ is a subset of $R$. If $\boldsymbol{y}$ is a point of $\varprojlim \boldsymbol{f}$ in $R$, then $y_2 \leq 1/4$ since $y_1 \in (1/8, 3/8)$. If $y_2 < 1/4$ then $y_3 > 7/8$ and $\boldsymbol{y}$ is not in $R$. So $y_2 = 1/4$ and we have $y_3 = 3/4$, thus $\boldsymbol{y} \in N$. Then, $N$ and $\varprojlim \boldsymbol{f} - N$ are mutually separated sets whose union is $\varprojlim \boldsymbol{f}$. □

We close this section with Theorem 2.12, a generalization of Theorem 3.2 in [3]. In [8, Theorem 2.4], although his emphasis is on

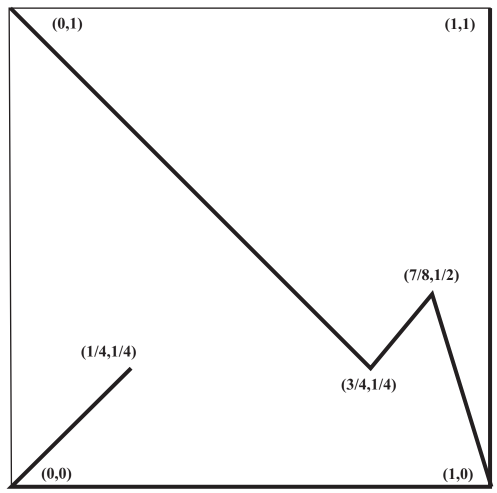


FIGURE 7. *Graph of the bonding function (Example 2.8)*

the decomposability of the inverse limit, Scott Varagona shows that under slightly different conditions than those of Theorem 2.13 that an upper semi-continuous function that is the union of two continuum valued functions on $[0,1]$ produces a continuum. If each of $f : X \to 2^X$ and $g : X \to 2^X$ is an upper semi-continuous function, we say that $f$ and $g$ have a *coincidence point* provided there is a point $t$ of $X$ such that $f(t) \cap g(t) \neq \emptyset$. We observe that two upper semi-continuous interval-valued functions on $[0,1]$, one of which is surjective, have a coincidence point; we leave the proof to the reader. Recall that we use $C(X)$ to denote the set of connected elements of $2^X$.

**Theorem 2.9.** *Suppose $f : [0,1] \to C([0,1])$ is a surjective upper semi-continuous function and $g : [0,1] \to C([0,1])$ is an upper semi-continuous function. Then $f$ and $g$ have a coincidence point.*

**Lemma 2.10.** *Suppose $X$ is a Hausdorff continuum and $f_i : X \to 2^X$ is an upper semi-continuous function for each positive integer $i$, $g : X \to 2^X$ is an upper semi-continuous function. Suppose further that $n$ is a positive integer such that $f_n$ and $g$ have a coincidence point and $f_i$ is surjective for $i > n$. If $\boldsymbol{\varphi}$ is a sequence of upper* *semi-continuous functions such that* $\varphi_i = f_i$ *for* $i \neq n$ *and* $\varphi_n = g$, *then* $\varprojlim \boldsymbol{f}$ *and* $\varprojlim \boldsymbol{\varphi}$ *have a point in common.*

*Proof:* There is a point $t$ of $X$ such that $f_n(t) \cap g(t) \neq \emptyset$. Since $f_i$ is surjective for each $i > n$, it is not difficult to see that for each $y \in f_n(t)$ there is a point $\boldsymbol{x}$ of $\varprojlim \boldsymbol{f}$ such that $x_n = y$ and $x_{n+1} = t$. If $y \in g(t) \cap f_n(t)$, the point $\boldsymbol{x}$ of $\varprojlim \boldsymbol{f}$ for which $x_n = y$ and $x_{n+1} = t$ is in $\varprojlim \boldsymbol{\varphi}$. □

**Definition 2.11.** Let $\mathcal{F}$ be a collection of upper semi-continuous functions from a compact Hausdorff space $X$ into $2^X$. A function $f : X \to 2^X$ is said to be *universal* with respect to $\mathcal{F}$ provided that for each $g \in \mathcal{F}$ there is a point $t$ of $X$ such that $f(t) \cap g(t) \neq \emptyset$.

**Theorem 2.12.** *If* $\mathcal{F}$ *is a collection of upper semi-continuous Hausdorff continuum-valued functions from a non-degenerate Hausdorff continuum* $X$ *into* $C(X)$, *one of which is surjective and universal with respect to* $\mathcal{F}$, *and* $f$ *is a closed subset of* $X \times X$ *that is the set theoretic union of the collection* $\mathcal{F}$, *then* $f : X \to 2^X$ *is an upper semi-continuous function and* $\varprojlim \boldsymbol{f}$ *is a Hausdorff continuum.*

*Proof:* Since $f$ is a closed subset of $X \times X$ and each point of $X$ is a first coordinate of some point of $f$, $f$ is upper semi-continuous, [4, Theorem 2.1]. Since $\varprojlim \boldsymbol{f}$ is compact, we need only to show that this inverse limit is connected. Suppose $f_1$ is a member of $\mathcal{F}$ that is surjective and universal with respect to $\mathcal{F}$. Since $f_1$ is surjective, the inverse limit, $\varprojlim \boldsymbol{f_1}$, is a non-degenerate continuum. Let $\boldsymbol{y}$ be a point of $\varprojlim \boldsymbol{f}$. There exists a sequence $\varphi_1, \varphi_2, \varphi_3, \dots$ such that $\varphi_i \in \mathcal{F}$ and $\varphi_i(y_{i+1}) = y_i$ for each positive integer $i$. Let $C_1 = \varprojlim \boldsymbol{f}_1$, and, if $n$ is a positive integer with $n > 1$, let $C_n$ be the inverse limit of the sequence $\varphi_1, \varphi_2, \dots, \varphi_{n-1}, f_1, f_1, f_1, \dots$. For each $n$, $C_n$ is a Hausdorff continuum by [4, Theorem 4.7] since $\varphi_1, \varphi_2, \dots, \varphi_{n-1}$, and $f_1$ are Hausdorff continuum valued. Since $f_1$ is surjective, by Lemma 2.10, $C_n \cap C_{n+1} \neq \emptyset$. Thus, $\bigcup_{i>0} C_i$ is connected. Moreover, for each $n$, since $f_1$ is surjective and $y_i \in \varphi_i(y_{i+1})$ for each $i$, there is a point $\boldsymbol{p^n}$ of $C_n$ such that $\pi_i(\boldsymbol{p^n}) = y_i$ for $i \leq n$. It follows that $\boldsymbol{y} \in \overline{C}$. Since $\varprojlim \boldsymbol{f}$ is the union of a collection of connected sets all containing the connected set $\varprojlim \boldsymbol{f_1}$, $\varprojlim \boldsymbol{f}$ is connected. □

Using Theorem 2.9, we have the following corollary to Theorem 2.12.

**Theorem 2.13.** *If $f : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous function that is the union of two upper semi-continuous interval-valued functions one of which is surjective, then* $\varprojlim \boldsymbol{f}$ *is connected.*

## 3. The full projection property

In inverse limits of mappings, closed subsets of the inverse limit are the inverse limit of their projections. This fails for inverse limits with upper semi-continuous bonding functions as may be seen by the following example.

**Example 3.1.** Let $f : [0,1] \to C(0,1])$ be the function given by $f(x) = [0,1]$ for each $x \in [0,1]$. Then $\varprojlim \boldsymbol{f}$ is the Hilbert cube while $M = \{\boldsymbol{x} \in \varprojlim \boldsymbol{f} \mid x_1 \in [0,1]$ and $x_{i+1} = x_i$ for each $i\}$ is an arc that is a closed proper subset of $\varprojlim \boldsymbol{f}$ such that $\pi_i(M) = [0,1]$ for each $i$.

Alexander N. Cornelius presents a very nice characterization of the compact subsets of an inverse limit that are inverse limits of their projections in [1]. However, his characterization requires a rather detailed knowledge of the inverse limit in order for it to be checked. So we pose the following problem.

**Problem 3.2.** *Find conditions on the bonding functions that ensure that closed subsets of the inverse limit are the inverse limit of their projections.*

Of course, it would be of interest to settle this problem in the case of an inverse limit with a single bonding function, even a single one on $[0,1]$. Another special case of interest would be to settle the problem in case our attention is restricted to which subcontinua are inverse limits of their projections. Yet another possible interesting variation on this problem would be to assume the bonding functions are Hausdorff continuum-valued. One of the main uses of the theorem in ordinary inverse limits is to conclude that if a closed subset of an ordinary inverse limit has each of its projections the entire factor space, then it is the entire inverse limit. So we pose a variation on Problem 3.2.

If $X_1, X_2, X_3, \ldots$ is a sequence of Hausdorff spaces and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function for each positive integer $i$, we shall say that $M = \varprojlim \boldsymbol{f}$ has the *full projection* *property* provided it is true that if $H$ is a closed and connected subset of $M$ such that $\pi_n(H) = X_n$ for infinitely many positive integers $n$, then $H = M$.

**Problem 3.3.** *Find conditions on the bonding functions that ensure that an inverse limit has the full projection property.*

Again, it would be of interest to settle this problem in the case of an inverse limit with a single bonding function (or for subcontinua of such inverse limits), even a single one on $[0, 1]$. It could also be interesting to restrict attention to inverse limit sequences such that $f_i : X_{i+1} \to C(X_i)$. In addition, a study of the full projection property on closed subsets of the inverse limit could prove useful (i.e., drop the connectedness requirement). Example 3.5 shows that not every inverse limit in which the bonding functions have connected values possesses the full projection property. On the other hand, the full projection property is not just a property of inverse limits of mappings, as may be seen from the following example. Later we use the fact that this example has the full projection property to show that it is an indecomposable continuum. See Theorem 4.3. It should also be mentioned that Varagona [8, Lemma 3.1] gives a sufficient condition for an inverse limit with a single bonding function to have the full projection property, although it cannot be used to show that Example 3.4 has the full projection property.

**Example 3.4.** Let $f : [0, 1] \to 2^{[0,1]}$ be the function whose graph consists of the union of three straight line segments, one from $(0, 0)$ to $(1/2, 1)$, one from $(1/2, 1)$ to $(1/2, 0)$, and one from $(1/2, 0)$ to $(1, 1)$. Then, $\varprojlim \boldsymbol{f}$ has the full projection property. (See Figure 8 for the graph of this function.)

*Proof:* Let $n$ be a positive integer and $G'_n = \{(x_1, x_2, \ldots, x_{n+1}) \in [0, 1]^{n+1} \mid x_i \in f(x_{i+1})$ for $1 \leq i \leq n\}$. We first show inductively that if $(p_1, p_2, \ldots, p_{n+1}) \in G'_n$ and $p_{n+1} \notin \{0, 1\}$, then $G'_n - \{(p_1, \ldots, p_{n+1})\}$ is the union of two mutually separated sets, one containing $(0, 0, \ldots, 0)$ and the other containing $(1, 1, \ldots, 1)$; i.e., $G'_n$ is an arc with end points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. Since $G'_1 = (G(f))^{-1}$ this is true for $n = 1$. Assume it is true for $n = k$ and let $p = (p_1, p_2, \ldots, p_{k+2})$ be a point of $G'_{k+1}$ such that $p_{k+2} \notin \{0, 1\}$. We consider cases: $p_{k+2} \neq 1/2$ and $p_{k+2} = 1/2$. If $0 < p_{k+2} < 1/2$, then $p_{k+1} \notin \{0, 1\}$ so $G'_k - \{(p_1, \ldots, p_{k+1})\} = A_{k,0} \cup A_{k,1}$

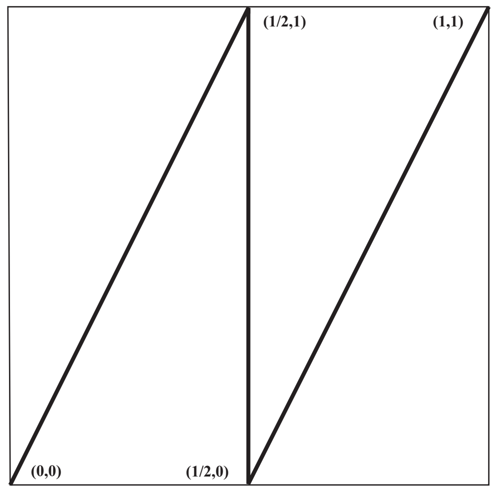


FIGURE 8. *Graph of the bonding function (Example 3.4)*

where $A_{k,0}$ and $A_{k,1}$ are mutually separated sets with $(0,0,\ldots,0) \in A_{k,0}$ and $(1,1,\ldots,1) \in A_{k,1}$. Let $A_{k+1,0} = \{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid x_{k+2} < 1/2$ and $(x_1,x_2,\ldots,x_{k+1}) \in A_{k,0}\}$ and let $A_{k+1,1} = \{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid (1)\, x_{k+2} \geq 1/2$ or $(2)\, x_{k+2} < 1/2$ and $(x_1,x_2,\ldots,x_{k+1}) \in A_{k,1}\}$. Then $(0,0,\ldots,0)$ is in $A_{k+1,0}$, $(1,1,\ldots,1)$ is in $A_{k+1,1}$, $A_{k+1,0}$ and $A_{k+1,1}$ are mutually separated, and $G'_{k+1} - \{p\} = A_{k+1,0} \cup A_{k+1,1}$. If $1/2 < p_{k+2} < 1$, we obtain the desired separation by letting $A_{k+1,0} = \{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid (1)\, x_{k+2} \leq 1/2$ or (2) $x_{k+2} > 1/2$ and $(x_1,x_2,\ldots,x_{k+1}) \in A_{k,0}\}$ and $A_{k+1,1} = \{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid x_{k+2} > 1/2$ and $(x_1,x_2,\ldots,x_{k+1}) \in A_{k,1}\}$. If $p_{k+2} = 1/2$, there are three possibilities: $p_{k+1} = 0$, $p_{k+1} = 1$, and $p_{k+1} \notin \{0,1\}$. Suppose $p_{k+1} = 0$. Note that $p = (0,0,\ldots,0,1/2)$. Let $A_{k+1,0} = \{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid x_{k+2} \leq 1/2\} - \{p\}$ and $A_{k+1,1} = \{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid x_{k+2} > 1/2\}$. From the observation that $p$ is the only limit point of $A_{k+1,1}$ having last coordinate $1/2$, it follows that $A_{k,0}$ and $A_{k,1}$ are mutually separated. The case that $p_{k+1} = 1$ is similar. If $p_{k+1} \notin \{0,1\}$, then $G'_k = A_{k,0} \cup A_{k,1}$ where $(0,0,\ldots,0) \in A_{k,0}$, $(1,1,\ldots,1) \in A_{k,1}$, and $A_{k,0}$ and $A_{k,1}$ are mutually separated. We obtain the desired separation by letting $A_{k+1,0} =$

$\{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid (1)\, x_{k+2} < 1/2$ or $(2)\ x_{k+2} = 1/2$ and $(x_1,x_2,\ldots,x_{k+1}) \in A_{k,1}\}$ and $A_{k+1,1} = \{(x_1,x_2,\ldots,x_{k+2}) \in G'_{k+1} \mid (1)\, x_{k+2} > 1/2$ or $(2)\ x_{k+2} = 1/2$ and $(x_1,x_2,\ldots,x_{k+1}) \in A_{k,0}\}$.

Now suppose $H$ is a subcontinuum of $\varprojlim \boldsymbol{f}$ such that $\pi_i(H) = [0,1]$ for infinitely many positive integers $i$. Let $\boldsymbol{p}$ be a point of $\varprojlim \boldsymbol{f}$ such that $\boldsymbol{p} \notin \{(0,0,0,\ldots),(1,1,1,\ldots)\}$. Suppose $n$ is a positive integer. There is a positive integer $m \geq n$ such that $p_{m+1} \notin \{0,1\}$ and $\pi_{m+1}(H) = [0,1]$. Then $(p_1,\ldots,p_{m+1})$ is in $G'_m$ and $G'_m - \{(p_1,\ldots,p_{m+1})\} = A_{m,0} \cup A_{m,1}$ where $(0,0,\ldots,0) \in A_{m,0}$ and $(1,1,\ldots,1) \in A_{m,1}$. Thus, $H$ intersects the two mutually separated sets $A_{m,0} \times [0,1]^\infty$ and $A_{m,1} \times [0,1]^\infty$ so $H$ contains a point in the boundary of each of them. Consequently, $H$ contains a point $\boldsymbol{q}$ such that $(q_1,q_2,\ldots,q_{m+1}) = (p_1,p_2,\ldots,p_{m+1})$. Therefore, $d(\boldsymbol{q},\boldsymbol{p}) < 2^{-(m+1)} < 2^{-n}$. It follows that $\boldsymbol{p} \in \overline{H}$ so $H = \varprojlim \boldsymbol{f}$. □

One conjecture we had regarding Problem 3.3 was that if $f : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous function such that the graph of $f$ is irreducible from $\{0\} \times [0,1]$ to $\{1\} \times [0,1]$, then $\varprojlim \boldsymbol{f}$ has the full projection property. However, this turns out not to be the case as may be seen from the following example.

**Example 3.5.** Let $f : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function whose graph consists of three straight line intervals, one from (0,0) to (1/2,1), one from (1/2,1) to (1/2,1/2), and the third from (1/2,1/2) to (1,1). Then $\varprojlim \boldsymbol{f}$ does not have the full projection property. (See Figure 9 for a picture of the bonding function.)

*Proof:* Let $M = \varprojlim \boldsymbol{f}$ and $A_0 = \{\boldsymbol{x} \in M \mid x_1 \in [0,1/2]$ and $x_{k+1} = x_k/2$ for each positive integer $k\}$. For each positive integer $n$, let $A_{2n-1} = \{\boldsymbol{x} \in M \mid x_1 \in [1/2,1]$ and $x_k = x_1$ for $1 \leq k \leq n$ while $x_{k+1} = x_k/2$ for $k > n\}$ and let $A_{2n} = \{\boldsymbol{x} \in M \mid x_1 \in [1/2,1]$ and $x_k = x_1$ for $1 \leq k \leq n$ while $x_k = 1/2^{k-n}$ for $k > n\}$. For each integer $i \geq 0$, $A_i$ is an arc. Let $\boldsymbol{p_0}$ denote the point of $M$ such that $\pi_k(\boldsymbol{p_0}) = 1/2^k$ for each positive integer $k$. For each positive integer $n$, let $\boldsymbol{p_{2n-1}}$ be the point of $M$ such that $\pi_k(\boldsymbol{p_{2n-1}}) = 1$ for $1 \leq k \leq n$ and $\pi_k(\boldsymbol{p_{2n-1}}) = 1/2^{k-n}$ for $k > n$; let $\boldsymbol{p_{2n}}$ be the point of $M$ such that $\pi_k(\boldsymbol{p_{2n}}) = 1/2$ for $1 \leq k \leq n$ and $\pi_k(\boldsymbol{p_{2n}}) = 1/2^{k-n}$ for $k > n$. Note that $A_i \cap A_{i+1} = \{\boldsymbol{p_i}\}$

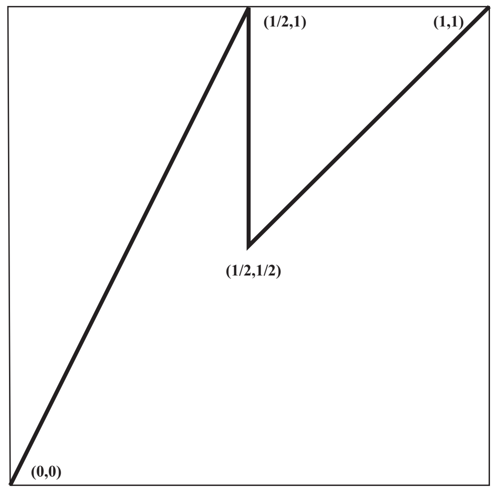


FIGURE 9. *Graph of the bonding function (Example 3.5)*

for each non-negative integer $i$. Then $R = \bigcup_{i \geq 0} A_i$ is a topological ray such that $\overline{R} - R = \{\boldsymbol{x} \in M \mid x_1 \in [1/2, 1]$ and $x_n = x_1$ for each positive integer $n\}$. Finally, $\overline{R}$ is a proper subcontinuum of $M$ since $\overline{R}$ does not contain the point $(1, 1/2, 1/2, 1/2, \dots)$ of $M$, but $\pi_i(\overline{R}) = [0, 1]$ for each positive integer $i$. □

## 4. INDECOMPOSABILITY

D. P. Kuykendall [5] gave a characterization of indecomposability of an inverse limit of a sequence of mappings. Numerous investigations of inverse limits with mappings have turned up sufficient conditions that the inverse limit be indecomposable. A number of these require that the bonding maps or compositions of the bonding maps satisfy an approximate two-pass condition. For inverse limits on intervals with mappings, the two-pass condition is a simple, sufficient condition for indecomposability of the inverse limit. That condition for maps of intervals is that there are two non-overlapping intervals each of which is mapped onto the entire interval. The corresponding theorem for upper semi-continuous functions on $[0, 1]$ fails miserably. For example, the Hilbert cube is the inverse limit of the single upper semi-continuous function $f : [0, 1] \to 2^{[0,1]}$ whose graph is the entire disk, $[0,1] \times [0,1]$. Varagona [8] has investigated decomposability and indecomposability in inverse limits with upper semi-continuous bonding functions and has obtained some very nice results. He has sufficient conditions for an inverse limit with certain upper semi-continuous bonding functions to be indecomposable and conditions that are sufficient that the inverse limit be decomposable. He has applied some of his results to show that the inverse limit of the function whose graph is the $\sin(1/x)$-curve compressed into $[0,1] \times [0,1]$ produces an indecomposable continuum. There seems to be ample reason to devote further study to ways indecomposability arises in inverse limits with set-valued functions.

**Problem 4.1.** *Find sufficient conditions on the bonding functions so that* $\varprojlim \boldsymbol{f}$ *is indecomposable.*

Once again, it would be of interest to obtain solutions to this problem for inverse limits with a single upper semi-continuous function, even one on $[0,1]$ (and it could be of interest to consider that $f_i : X_{i+1} \to C(X_i)$ for each $i$). We provide one such theorem, see Theorem 4.3, that does not require that the bonding functions have connected values to close this section. We offer the following replacement for the two-pass condition. If $X$ and $Y$ are Hausdorff continua and $f : X \to 2^Y$ is an upper semi-continuous function, we say that $f$ satisfies the *two-pass condition* if there are mutually exclusive connected open subsets $U$ and $V$ of $X$ so that $f \mid U$ and $f \mid V$ are mappings and $\overline{f(U)} = \overline{f(V)} = Y$. If $n \geq 3$ is an integer, we say that the continuum $T$ is a simple $n$-od provided there is a point $J$ of $T$ such that $T$ is the union of $n$ arcs, each two of which intersect only at $J$. We call $J$ the *junction point* of $T$ and the other end points of the arcs that make up $T$ the *end points* of $T$.

**Lemma 4.2.** *Suppose $T$ is an arc or a simple $n$-od for some integer $n \geq 3$ and $T$ is the union of two proper subcontinua $H$ and $K$. If $U$ and $V$ are mutually exclusive connected open subsets of $T$, then one of $U$ and $V$ is a subset of one of $H$ and $K$.*

*Proof:* If $T$ is an arc, let $J$ denote a separating point of $T$, and let $J$ be the junction point of $T$ if $T$ is an $n$-od. The point $J$ cannot belong to both $U$ and $V$. Suppose $J \notin U$ and $A$ is the end point of $T$ such that $U$ is a subset of the arc $[J, A]$. Assume $A \in H$. If $J \in H$, then $U \subseteq H$. If $J \notin H$ and $U$ is not a subset of either $H$

or $K$, then $H \cap K \subseteq U$. If $A \in U$, then $T - U \subseteq K$ so $V \subseteq K$. If $A \notin U$, then $T - U$ is the union of two mutually exclusive closed sets $C$ and $D$ with $A \in C$. Then $V \subseteq C$ or $V \subseteq D$. Since $C \subseteq H$ and $D \subseteq K$, $V \subseteq H$ or $V \subseteq K$. □

One consequence of our next theorem is that the inverse limit from Example 3.4 is indecomposable. This theorem generalizes [2, Theorem 3.4]. In private correspondence with the author, Varagona observed that if $f : [0,1] \to 2^{[0,1]}$ is the upper semi-continuous function whose graph consists of three straight line intervals, one from $(0,1)$ to $(0,0)$, one from $(0,0)$ to $(1/2,1)$, and one from $(1/2,1)$ to $(1,0)$, then $f$ satisfies the two-pass condition, but $\varprojlim \boldsymbol{f}$ does not have the full projection property.

**Theorem 4.3.** *Suppose $T_1, T_2, T_3, \ldots$ is a sequence such that if $i$ is a positive integer, then $T_i$ is an arc or there is a positive integer $n_i$ so that $T_i$ is a simple $n_i$-od. If $f_i : T_{i+1} \to 2^{T_i}$ is an upper semi-continuous function satisfying the two-pass condition for each positive integer $i$ and $\varprojlim \boldsymbol{f}$ is a continuum with the full projection property, then $\varprojlim \boldsymbol{f}$ is indecomposable.*

*Proof:* Suppose $M = \varprojlim \boldsymbol{f}$ is the union of two proper subcontinua $H$ and $K$. Since $M$ has the full projection property, there is a positive integer $n$ such that if $m \geq n$, then $\pi_m(H) \neq T_m$ and $\pi_m(K) \neq T_m$. Since $f_n$ satisfies the two pass condition there are mutually exclusive connected open subsets $U$ and $V$ of $T_{n+1}$ so that $f_n \mid U$ and $f_n \mid V$ are mappings and $\overline{f_n(U)} = \overline{f_n(V)} = T_n$. Since $\pi_{n+1}(H)$ and $\pi_{n+1}(K)$ are two subcontinua whose union is $T_{n+1}$, by Lemma 4.2, one of $U$ and $V$ is a subset of one of $\pi_{n+1}(H)$ and $\pi_{n+1}(K)$. Suppose $U \subseteq \pi_{n+1}(H)$. If $t \in f_n(U)$ there is a point $s$ of $U$ such that $f_n(s) = t$. Since $U \subseteq \pi_{n+1}(H)$, there is a point $\boldsymbol{x}$ of $H$ such that $x_{n+1} = s$. Then $x_n = t$ so $t \in \pi_n(H)$ and it follows that $f_n(U) \subseteq \pi_n(H)$. However, since $\overline{f_n(U)} = T_n$, this contradicts the fact that $\pi_n(H) \neq T_n$. The other possibilities similarly lead to a contradiction. □

## 5. Subsequence theorem

If $\boldsymbol{X}$ is a sequence of compact Hausdorff spaces and $\boldsymbol{f}$ is a sequence of mappings such that $f_i : X_{i+1} \to X_i$ for each positive integer $i$, and $m$ and $n$ are integers with $n < m$, it is sometimes convenient to denote by $f_{nm}$ the mapping from $X_m$ into $X_n$ given by $f_{nm} = f_n \circ f_{n+1} \circ \cdots \circ f_{m-1}$. One very powerful tool in the theory of ordinary inverse limits is the subsequence theorem; i. e., if $\boldsymbol{n}$ is an increasing sequence of positive integers and $\boldsymbol{g}$ is a sequence of mappings such that $g_i = f_{n_i\, n_{i+1}}$ for each positive integer $i$, then $\varprojlim \boldsymbol{f}$ and $\varprojlim \boldsymbol{g}$ are homeomorphic. Early in the development of the theory of inverse limits with set-valued functions came the observation that the subsequence theorem does not carry over. For examples of this phenomenon see [4, Example 3, p. 127 and Example 4, p. 128]. It would be nice to have some sufficient conditions on the bonding functions to ensure that the subsequence theorem holds. Employing similar notation for $f_{nm}$ for upper semi-continuous functions, we state the following problem.

**Problem 5.1.** *If $\boldsymbol{X}$ is a sequence of compact Hausdorff spaces and $\boldsymbol{f}$ is a sequence of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ (or $f_i : X_{i+1} \to C(X_i)$) for each positive integer $i$, and $n_1, n_2, n_3, \ldots$ is an increasing sequence of positive integers, find sufficient conditions on the bonding functions such that if $g_i = f_{n_i\, n_{i+1}}$ for each $i$, then $\varprojlim \boldsymbol{f}$ and $\varprojlim \boldsymbol{g}$ are homeomorphic.*

Like some of the other problems stated in this paper, Problem 5.1 may be too general to attack at the present. A simpler and, perhaps, more tractable problem is its counterpart for inverse limits with a single set-valued function.

**Problem 5.2.** *If $f : X \to 2^X$ is an upper semi-continuous function on a compact Hausdorff space $X$ and $n$ is a positive integer greater than 1, find sufficient conditions such that $\varprojlim \boldsymbol{f}$ is homeomorphic to $\varprojlim \boldsymbol{f^n}$.*

Of course, an answer to this problem for $n = 2$ would be of interest as would an answer in case $X = [0, 1]$. One simple, but obvious, such condition for inverse limits with a single set-valued function is that the bonding function $f$ be idempotent, i. e., $f^2 = f$. This is the case with the set-valued function that is the union of the identity, $id$, and $1 - id$ on $[0, 1]$.

**Theorem 5.3.** *If $f : X \to 2^X$ is an upper semi-continuous function on a compact Hausdorff space such that $f^2 = f$ and $n$ is a positive integer, then $\varprojlim \boldsymbol{f}$ and $\varprojlim \boldsymbol{f^n}$ are homeomorphic.*

## 6. Incommensurate objects

Nall [7] has shown that the 2-cell is not homeomorphic to an inverse limit with a single upper semi-continuous function $f : [0, 1] \to 2^{[0,1]}$. At the meeting of the American Mathematical Society at Baylor University in October 2009, Alejandro Illanes announced that a simple closed curve also cannot be such an inverse limit. Many opportunities exist for further exploration along these lines.

**Problem 6.1.** *For a given compact Hausdorff space $X$, which compact sets are not homeomorphic to inverse limits with a single upper semi-continuous function $f : X \to 2^X$ $(f : X \to C(X))$?*

Of course, adding to the list of known examples for inverse limits on $[0, 1]$ would be of considerable interest. It would also be of interest to study such problems for inverse limits of upper semi-continuous functions on the unit circle or the simple triod.

## 7. Bases for the topology

In ordinary inverse limits, $\{\pi_i^{-1}(O) \mid i$ is a positive integer and $O$ is open in $X_i\}$ is a basis for the topology of the inverse limits. However, for inverse limits with upper semi-continuous bonding functions, this is not always the case. We may see this from Example 3.1. Let $R = [0, 1/4) \times (3/4, 1] \times \mathcal{Q}$. Suppose $i$ is a positive integer and $O$ is an open subset of $[0, 1]$. If $t$ is a point of $O$, then $(t, t, t, \dots)$ is a point of the inverse limit in $\pi_i^{-1}(O)$, but since $t$ cannot belong to both $[0, 1/4)$ and $(3/4, 1]$, $(t, t, t, \dots)$ is not in $R$. This leads us the our next problem. This problem may or may not be very interesting, but it fits our theme so we include it.

**Problem 7.1.** *Find conditions on the bonding functions that ensure that $\{\pi_i^{-1}(O) \mid i$ is a positive integer and $O$ is open in $X_i\}$ is a basis for the topology of the inverse limit with upper semi-continuous bonding functions that are not mappings.*

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