

CONCERNING DIMENSION AND TREE-LIKENESS OF INVERSE LIMITS WITH SET-VALUED FUNCTIONS

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ABSTRACT. From a theorem of Van Nall it is known that inverse limits with sequences of upper semi-continuous set-valued functions with 0-dimensional values have dimension bounded by the dimensions of the factor spaces. Information is also available about the dimension of inverse limits with sequences of upper semi-continuous continuum-valued bonding functions having graphs that are mappings on the factor spaces that have continua appended at each point of a closed set, however the conclusion of this theorem allows the possibility of an infinite dimensional inverse limit. In this paper we show that inverse limits with sequences of certain surjective upper semi-continuous continuum-valued bonding functions have dimension bounded by the dimensions of the factor spaces. One consequence of our investigation is that certain inverse limits on $[0, 1]$ with upper semi-continuous continuum-valued functions are tree-like including those that are inverse limits on $[0, 1]$ with a single interval-valued bonding function that has no flat spots.

1. INTRODUCTION

In a list of problems in *An Introduction to Inverse Limits with Set-valued Functions* the author asked for sufficient conditions on bonding functions on $[0, 1]$ so that the inverse limit is a tree-like continuum [5, Problem 6.49]. In a recent article [6], the author showed that such an inverse limit is tree-like in the case that there is only one surjective bonding function f that is the union of two mappings on $[0, 1]$ having only one point (x, x) in common and $f^{-1}(x) = \{x\}$. Theorem 4.2 of this paper demonstrates tree-likeness of the inverse limit of certain sequences of surjective interval-valued functions on $[0, 1]$. One result of our investigation is that if $f : [0, 1] \rightarrow C([0, 1])$ is a surjective upper semi-continuous function and

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$G(f)$ contains no horizontal interval (i.e., f has no “flat spots”), then $\varprojlim \mathbf{f}$ is tree-like. We include examples showing the necessity of the hypotheses in our theorems and provide a sequence of new examples relating to the need for care with flat spots. We also include an example of a surjective bonding function $f : [0, 1] \rightarrow C([0, 1])$ that has flat spots but such that $\varprojlim \mathbf{f}$ is tree-like. Our tree-likeness results are obtained from a study of dimension in inverse limits on continua with continuum-valued bonding functions in Section 3.

2. DEFINITIONS

If X is a metric space, we denote the collection of closed subsets of X by 2^X and the connected elements of 2^X by $C(X)$. If X and Y are metric spaces, a function $f : X \rightarrow 2^Y$ is said to be *upper semi-continuous at the point x of X* provided that if V is an open set in Y that contains $f(x)$ then there is an open set U in X containing x such that if t is a point of U then $f(t) \subseteq V$. A function $f : X \rightarrow 2^Y$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of X . If $f : X \rightarrow 2^Y$, we say that f is *surjective* provided that for each $y \in Y$ there is a point $x \in X$ such that $y \in f(x)$. If $f : X \rightarrow 2^Y$ is a set-valued function, by the *graph* of f , denoted $G(f)$, we mean $\{(x, y) \in X \times Y \mid y \in f(x)\}$. It is known that if X and Y are compact and M is a subset of $X \times Y$ such that X is the projection of M to its set of first coordinates then M is closed if and only if M is the graph of an upper semi-continuous function [7, Theorem 2.1]. One consequence of this theorem is that if A is a closed subset of Y then $f^{-1}(A)$ is a closed subset of X . In the case that f is upper semi-continuous and single-valued, i.e., $f(x)$ is degenerate for each $x \in X$, f is a continuous function. By a *mapping* we mean a continuous function.

We denote by \mathbb{N} the set of positive integers. If $\mathbf{s} = s_1, s_2, s_3, \dots$ is a sequence (or $\mathbf{s} = s_1, s_2, \dots, s_n$ is a finite sequence), we normally denote the sequence in boldface type and its terms in italics. If \mathbf{X} is a sequence of metric spaces, we denote the product of the terms of \mathbf{X} by $\prod_{i>0} X_i$. The points of $\prod_{i>0} X_i$ are sequences so if $\mathbf{x} \in \prod_{i>0} X_i$, it should not be a problem denoting \mathbf{x} by x_1, x_2, x_3, \dots . However, we adopt the usual convention of enclosing the terms of \mathbf{x} in parentheses, $\mathbf{x} = (x_1, x_2, x_3, \dots)$, to signify that \mathbf{x} is a point of the product space. A metric d compatible with the product topology for $\prod_{i>0} X_i$ is given by $d(\mathbf{x}, \mathbf{y}) = \sum_{i>0} d_i(x_i, y_i)/2^i$ where, for each $i \in \mathbb{N}$, d_i is a metric for X_i bounded by 1.

Suppose \mathbf{X} is a sequence of metric spaces and \mathbf{f} is a sequence of upper semi-continuous functions such that $f_n : X_{n+1} \rightarrow 2^{X_n}$ for each $n \in \mathbb{N}$. Such a pair of sequences $\{\mathbf{X}, \mathbf{f}\}$ is called an *inverse limit sequence*. The *inverse limit* of the inverse limit sequence $\{\mathbf{X}, \mathbf{f}\}$, denoted $\varprojlim \mathbf{f}$, is the subset of $\prod_{i>0} X_i$ that contains the point (x_1, x_2, x_3, \dots) if and only if $x_n \in f_n(x_{n+1})$ for each positive integer n . In the case that f_n is a mapping, the condition $x_n \in f_n(x_{n+1})$ becomes $x_n = f_n(x_{n+1})$. For an inverse limit sequence $\{\mathbf{X}, \mathbf{f}\}$, the spaces X_i are called *factor spaces* and the functions f_n are called *bonding functions*. In the case that the sequences \mathbf{X} and \mathbf{f} are constant, the inverse limit sequence is said to be an inverse limit sequence with only one bonding function. That inverse limits are nonempty and compact when the factor spaces are compact and the bonding functions upper semi-continuous is [7, Theorem 3.2]; inverse limit spaces are metric because the product space that contains them is metric. If each of X, Y , and Z is a metric space and each of $f : X \rightarrow 2^Y$ and $g : Y \rightarrow 2^Z$ is a function, by $g \circ f : X \rightarrow 2^Z$ we mean the function given by $g \circ f(x) = \{z \in Z \mid \text{there is a point } y \in Y \text{ such that } y \in f(x) \text{ and } z \in g(y)\}$. If X is a metric space, we denote the identity on X by Id_X . If $\{\mathbf{X}, \mathbf{f}\}$ is an inverse limit sequence, we adopt the usual convention that if i and j are integers with $i < j$, $f_{ij} : X_j \rightarrow X_i$ is the function given by $f_{ij} = f_i \circ f_{i+1} \circ \dots \circ f_{j-1}$ while $f_{ii} = Id_{X_i}$.

Suppose \mathbf{X} is a sequence of metric spaces and $A \subseteq \mathbb{N}$. Denote by $p_A : \prod_{i>0} X_i \rightarrow \prod_{i \in A} X_i$ the *projection* given by $p_A(\mathbf{x}) = \mathbf{y}$ where $y_i = x_i$ for each $i \in A$. If $A = \{n\}$, we denote p_A by p_n . For inverse limits, we denote by π_A the restriction of p_A to $\varprojlim \mathbf{f}$.

By a *continuum* we mean a compact, connected metric space. If $\{\mathbf{X}, \mathbf{f}\}$ is an inverse limit sequence where each X_n is a continuum and each f_n is an upper semi-continuous set-valued function, we let $G'(f_1, f_2, \dots, f_n) = \{\mathbf{x} \in \prod_{i=1}^{n+1} X_i \mid x_j = f_j(x_{j+1}) \text{ for } 1 \leq j \leq n\}$. Sometimes we denote $G'(f_1, f_2, \dots, f_n)$ by G'_n . In the case that $n = 1$, $G'(f_1) = (G(f_1))^{-1}$. In many proofs of properties of inverse limits, the approximations of the inverse limit $G_n = G'_n \times (\prod_{i>n+1} X_i)$ are used extensively (e.g., [7, Theorem 3.2]).

We make use of the following theorem from [7] where it proved in a more general setting (Theorem 4.7, page 124).

Theorem 2.1. *Suppose \mathbf{X} is a sequence of continua and \mathbf{f} is a sequence of upper semi-continuous functions such that $f_n : X_{n+1} \rightarrow C(X_n)$ for each $n \in \mathbb{N}$. Then, $\varprojlim \mathbf{f}$ is a continuum.*

We use *dimension* in the standard sense as found in Hurewicz and Wallman [3]. If M is a compactum, $\dim(M)$ denotes its dimension. A continuum is *tree-like* provided it is homeomorphic to an inverse limit on trees (or, equivalently, its dimension is 1 and every mapping of it to a finite graph is inessential). Also, a continuum is tree-like if and only if it is a 1-dimensional continuum of trivial shape.

3. DIMENSION

In [9, Theorem 5.3] Van Nall proves Theorem 3.1 below and in [1, Theorem 1.2] Iztok Banic includes Theorem 3.2.

Theorem 3.1. (Nall) *Suppose m is a positive integer, \mathbf{X} is a sequence of compact metric spaces such that $\dim(X_n) \leq m$ for each $n \in \mathbb{N}$, and \mathbf{f} is a sequence of upper semi-continuous set-valued functions such that $f_n : X_{n+1} \rightarrow 2^{X_n}$ for each n . If $\dim(f_n(x)) = 0$ for each point $x \in X_{n+1}$ and each $n \in \mathbb{N}$, then $\dim(\varprojlim \mathbf{f}) \leq m$.*

Theorem 3.2. (Banic) *Suppose X is a nondegenerate continuum, A is a closed subset of X , and $g : X \rightarrow X$ is a mapping. If $f : X \rightarrow X$ is the upper semi-continuous function such that $G(f) = g \cup (A \times X)$, then $\dim(\varprojlim \mathbf{f}) \in \{\dim(X), \infty\}$.*

In this section, we prove a theorem (Theorem 3.4) that allows us to conclude that $\dim(\varprojlim \mathbf{f}) = \dim(X)$ under certain conditions on the set where the bonding functions have continuum values. In this sense our result is kin to that of Banic. Our theorem also allows sequences of factor spaces with sequences of bonding functions in the manner of Nall's theorem but allows function values to be nondegenerate continua instead of requiring that they be totally disconnected. Moreover, unlike Banic's Theorem, we do not require that the bonding functions be based on mappings of the factor spaces.

We begin with a theorem on the dimension of $G'(f_1, f_2, \dots, f_n)$. Its proof is embedded in the proof of Theorem 181 of [8]. We include its proof here for completeness. As is often the case for the study of dimension in inverse limits, we use covering dimension [3, Chapter V].

Theorem 3.3. *Suppose \mathbf{X} is a sequence of compact metric spaces and \mathbf{f} is a sequence of functions such that $f_n : X_{n+1} \rightarrow 2^{X_n}$ is upper semi-continuous for*

each positive integer n . If $\dim(G'(f_1, f_2, \dots, f_n)) \leq m$ for each positive integer n , then $\dim(\varprojlim \mathbf{f}) \leq m$.

PROOF. Let d_i be a metric on X_i that is bounded by 1 or each $i \in \mathbb{N}$ and d be usual metric on $\prod_{i>0} X_i$. For convenience, we use the metric ρ_j on $\prod_{i=1}^j X_i$ given by $\rho_j(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^j d_i(x_i, y_i)/2^i$. For each positive integer n , let $G_n = G'(f_1, f_2, \dots, f_n) \times \prod_{i>n+1} X_i$ and note that $\varprojlim \mathbf{f} \subseteq G_n$.

Suppose $\varepsilon > 0$. There is a positive integer n such that $\sum_{i>n} 1/2^i < \varepsilon/3$. There is a finite open cover \mathcal{U} of $G'(f_1, f_2, \dots, f_n)$ of order not greater than m and such that if $u \in \mathcal{U}$ then $\text{diam}(u) < \varepsilon/2$ (i.e., the mesh of \mathcal{U} is less than $\varepsilon/2$). If $u \in \mathcal{U}$ and \mathbf{x} and \mathbf{y} are in $p_{\{1,2,\dots,n+1\}}^{-1}(u)$, then $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n+1} d_i(x_i, y_i)/2^i + \sum_{i>n+1} d_i(x_i, y_i)/2^i$. Because $\text{diam}(u) < \varepsilon/2$, $\rho_{n+1}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n+1} d_i(x_i, y_i)/2^i < \varepsilon/2$. Therefore, $\text{diam}(p_{\{1,2,\dots,n+1\}}^{-1}(u)) < \varepsilon$. The order of $\mathcal{V} = \{p_{\{1,2,\dots,n+1\}}^{-1}(u) \mid u \in \mathcal{U}\}$ is not greater than m . Thus, we have an open cover of G_n of mesh less than ε and order not greater m .

Because $\varprojlim \mathbf{f} \subseteq G_n$ for each positive integer n , it follows that $\dim(\varprojlim \mathbf{f}) \leq m$. \square

Lemma 3.1. *Suppose m is a positive integer, \mathbf{X} is a sequence of continua such that $\dim(X_n) \leq m$ for each $n \in \mathbb{N}$, and \mathbf{O} is a sequence of open sets such that $O_n \subseteq X_n$ for each n . If \mathbf{g} is a sequence of mappings such that $g_n : O_{n+1} \rightarrow O_n$ for each n , then $\dim(G'(g_1, g_2, \dots, g_i)) \leq m$ for each positive integer i .*

PROOF. Suppose $i \in \mathbb{N}$. The function $\varphi : O_{i+1} \rightarrow G'(g_1, g_2, \dots, g_i)$ given by $\varphi(x) = (g_{1i+1}(x), \dots, g_{ii+1}(x), x)$ for each $x \in O_{i+1}$ is a 1-1 mapping and $\varphi(O_{i+1}) = G'(g_1, g_2, \dots, g_i)$. Because O_{i+1} is open in the continuum X_{i+1} it is the union of a sequence \mathbf{F} of compact sets. Because $\dim(\varphi(F_j)) \leq m$ for each j and φ is a homeomorphism on F_j , we see that $\varphi(O_{i+1})$ is the union of a sequence of compacta of dimension not greater than m . By [3, Theorem III 2], $\dim(\varphi(O_{i+1})) \leq m$ so $\dim(G'(g_1, g_2, \dots, g_i)) \leq m$. \square

Theorem 3.4. *Suppose m is a positive integer, \mathbf{X} is a sequence of continua of dimension not greater than m , and \mathbf{f} is a sequence of upper semi-continuous functions such that $f_n : X_{n+1} \rightarrow C(X_n)$ for each $n \in \mathbb{N}$. If, for each $n > 1$, Z_n is a closed 0-dimensional subset of X_n such that $g_n = f_n|_{(X_{n+1} - Z_{n+1})}$ is a mapping and $f_{ij}^{-1}(Z_i)$ is zero dimensional for each $i \geq 2$ and each $j > i$, then $\dim(\varprojlim \mathbf{f}) \leq m$.*

PROOF. Let $W_2 = Z_2$ and, for $n > 2$, let $W_n = Z_n \cup f_{n-1}^{-1}(W_{n-1})$. Note that $W_3 = Z_3 \cup f_2^{-1}(Z_2)$ and, in general, for $n > 2$, $W_n = Z_n \cup f_{n-1}^{-1}(Z_{n-1}) \cup \cdots \cup f_2^{-1}(Z_2)$. Moreover, because each bonding function is upper semi-continuous, W_n is closed for each $n \geq 2$. Suppose $n \geq 2$. Because Z_n and $f_{i_n}^{-1}(Z_i)$ are 0-dimensional for $2 \leq i \leq n-1$, $\dim(W_n) = 0$ [3, Theorem III 2].

Let $O_1 = X_1$ and, for $n > 1$, let $O_n = X_n - W_n$; O_n is open for each n . For each positive integer n , let $g_n = f_n|_{O_{n+1}}$. Because $O_{n+1} \subseteq X_{n+1} - Z_{n+1}$, g_n is a mapping for each n . If $t \in O_{n+1}$, then $t \notin f_{i_{n+1}}^{-1}(Z_i)$ for $1 \leq i \leq n$ so $f_n(t) \notin Z_n$ and $f_n(t) \notin f_{i_n}^{-1}(Z_i)$ for $1 \leq i \leq n-1$. It follows that $g_n(t) \in O_n$ and, thus, g_n is a mapping of O_{n+1} into O_n . By Lemma 3.1, $\dim(G'(g_1, g_2, \dots, g_n)) \leq m$ for each n .

Next we show that, for each $n \in \mathbb{N}$,

$$(*) \quad G'(f_1, f_2, \dots, f_n) \subseteq G'(g_1, g_2, \dots, g_n) \cup \\ (X_1 \times W_2 \times W_3 \times \cdots \times W_{n+1}) \cup (G'(f_1) \times W_3 \times W_4 \cdots \times W_{n+1}) \cup \\ (G'(f_1, f_2) \times W_4 \times \cdots \times W_{n+1}) \cup \cdots \cup (G'(f_1, f_2, \dots, f_{n-1}) \times W_{n+1}).$$

To that end, let $\mathbf{x} \in G'(f_1, f_2, \dots, f_n) - G'(g_1, g_2, \dots, g_n)$. Then there is an integer j , $1 \leq j \leq n$, such that $x_j \neq g_j(x_{j+1})$. If $j = n$, then $x_{n+1} \in W_{n+1}$ and $\mathbf{x} \in G'(f_1, f_2, \dots, f_{n-1}) \times W_{n+1}$. If $1 < j < n$ and $x_k = g_k(x_{k+1})$ for $j < k \leq n$ then $x_{j+1} \in W_{j+1}$ and $x_{i+1} \in g_i^{-1}(x_i)$ for $j+1 \leq i \leq n$. In this case $\mathbf{x} \in G'(f_1, f_2, \dots, f_{j-1}) \times W_{j+1} \times \cdots \times W_{n+1}$. If $x_1 \neq g(x_2)$ but $x_k = g_k(x_{k+1})$ for $2 \leq k \leq n$, then $\mathbf{x} \in X_1 \times W_2 \times \cdots \times W_{n+1}$.

With (*) established, we proceed by induction to prove that $\dim(G'(f_1, f_2, \dots, f_n)) \leq m$ for each $n \in \mathbb{N}$. Note that $\dim(G'(f_1)) \leq m$ because $G'(f_1) \subseteq G'(g_1) \cup (X_1 \times W_2)$ and $\dim(G'(g_1)) \leq m$ by Lemma 3.1 while $\dim(X_1 \times W_2) \leq m$ by [3, Corollary, p. 33] being the product of a set of dimension not greater than m with a 0-dimensional set. Suppose inductively that $\dim(G'(f_1, f_2, \dots, f_i)) \leq m$ for $1 \leq i < k$. Then, again using the Corollary on p. 33 of [3], we see that $X_1 \times W_2 \times W_3 \times \cdots \times W_{k+1}$, $G'(f_1) \times W_3 \times W_4 \cdots \times W_{k+1}$, $G'(f_1, f_2) \times W_4 \times \cdots \times W_{k+1}, \dots, G'(f_1, f_2, \dots, f_{k-1}) \times W_{k+1}$ all have dimension not greater than m . Because $\dim(G'(g_1, g_2, \dots, g_k)) \leq m$ it follows from (*), the inductive hypothesis, and Theorem III 2 of [3] that $\dim(G'(f_1, f_2, \dots, f_k)) \leq m$ being a subset of an m -dimensional space.

We now apply Theorem 3.3 to see that $\dim(\varprojlim \mathbf{f}) \leq m$. □

4. TREE-LIKENESS

We now turn to showing that certain inverse limits on $[0, 1]$ with interval-valued functions are tree-like. Our theorem depends on a theorem of Włodzimierz Charatonik and Robert P. Roe [2] concerning the shape of inverse limits with set-valued functions, Theorem 4.1 below. However, beyond Theorem 4.1, we do not need much information about shape other than the fact that among continua of dimension 1, the property of having trivial shape is equivalent to being tree-like.

Theorem 4.1. (Charatonik and Roe) *Suppose \mathbf{X} is a sequence of finite dimensional continua with trivial shape and \mathbf{f} is a sequence of functions such that $f_n : X_{n+1} \rightarrow C(X_n)$ is an upper semi-continuous function for each $n \in \mathbb{N}$. If $f_n(x)$ has trivial shape for each $n \in \mathbb{N}$ and each $x \in X_{n+1}$, then $\varprojlim \mathbf{f}$ has trivial shape.*

Because in compact metric spaces being totally disconnected is equivalent to being 0-dimensional [3, Prop. D, p. 22] the following theorem is an immediate consequence of Theorems 3.4 and 4.1.

Theorem 4.2. *Suppose \mathbf{f} is a sequence of functions such that $f_n : [0, 1] \rightarrow C([0, 1])$ is a surjective upper semi-continuous function for each positive integer n . If, for each $n > 1$, Z_n is a closed totally disconnected subset of $[0, 1]$ such that if $f_n(t)$ is nondegenerate then $t \in Z_n$ and $f_{i_n}^{-1}(Z_i)$ is totally disconnected for each i , $1 \leq i \leq n$, then $\varprojlim \mathbf{f}$ is a tree-like continuum.*

PROOF. Let $M = \varprojlim \mathbf{f}$; because each bonding function is continuum-valued, M is a continuum by Theorem 2.1. By Theorem 3.4, $\dim(M) \leq 1$; because each bonding function is surjective M is easily seen to be nondegenerate so $\dim(M) = 1$. By Theorem 4.1, M has trivial shape. Because M is 1-dimensional, M is a tree-like continuum. \square

In the remainder of this section, we make use of the following theorem of Nall [9, Theorem 5.4].

Theorem 4.3. (Nall) *Suppose X_1 is a continuum such that every nondegenerate subcontinuum K of X_1 contains a countable set that separates K . If \mathbf{X} is a sequence of compacta and \mathbf{f} is a sequence of upper semi-continuous functions such that, for each positive integer n , $f_n : X_{n+1} \rightarrow 2^{X_n}$ and, for each $y \in X_n$, $\dim(f_n^{-1}(y)) = 0$, then $\dim(\varprojlim \mathbf{f}) \leq 1$.*

In correspondence with the author Nall suggested the following consequence of Theorem 4.3. The reader should note that the bonding functions are assumed to

be continuum-valued. Examples show that without this hypothesis, the statement is false, e.g., see Examples 4.1 and 4.2 of [6].

Theorem 4.4. *Suppose \mathbf{X} is a sequence of trees and \mathbf{f} is a sequence of upper semi-continuous functions such that $f_n : X_{n+1} \rightarrow C(X_n)$ for each $n \in \mathbb{N}$. If $\dim(f_n^{-1}(x)) = 0$ for each $x \in X_n$ and each positive integer n , then $\varprojlim \mathbf{f}$ is a tree-like continuum.*

PROOF. Let $M = \varprojlim \mathbf{f}$; M is a continuum by Theorem 2.1. By Theorem 4.3, $\dim(M) \leq 1$. Because f is surjective, M is nondegenerate so $\dim(M) = 1$. By Theorem 4.1, M has trivial shape so M is tree-like. \square

If $f : [0, 1] \rightarrow 2^{[0,1]}$ is a function, we say that f has a *flat spot* provided $G(f)$ contains a nondegenerate horizontal interval. Corollary 4.1 below would be a corollary of Theorem 4.2 if we were to add the hypothesis that the closure of the set of points where f has interval values is totally disconnected. However, the statement is a corollary of Theorem 4.4.

Corollary 4.1. *If $f : [0, 1] \rightarrow C([0, 1])$ is a surjective upper semi-continuous function with no flat spots, then $\varprojlim \mathbf{f}$ is a tree-like continuum.*

5. EXAMPLES

Without some conditions limiting the role of flat spots on the graph of an upper semi-continuous function $f : [0, 1] \rightarrow C([0, 1])$, our dimension results fail. We begin with a couple of well-known examples. In our first example, the set $Z = \{t \in [0, 1] \mid f(t) \text{ is nondegenerate}\} = \{0\}$ but $f^{-1}(0) = [0, 1]$. The inverse limit is infinite dimensional because it contains $([0, 1] \times 0)^\infty$.

Example 5.1. *Let $f : [0, 1] \rightarrow C([0, 1])$ be the function given by $f(t) = 0$ if $t \neq 0$ and $f(0) = [0, 1]$. Then, $\varprojlim \mathbf{f}$ is infinite dimensional.*

In our next example the set $Z = \{t \in [0, 1] \mid f(t) \text{ is nondegenerate}\} = \{1/2, 1\}$ but $f^{-1}(1/2) = [1/2, 1]$. The inverse limit contains $[0, 1/2] \times \{1/2\} \times [1/2, 1] \times \{1\}^\infty$ so it is not 1-dimensional. (It is known that the inverse limit is a 2-cell with a sticker [8, Example 139, p. 104] or [7, Example 5]).

Example 5.2. *Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = 0$ for $0 \leq t < 1/2$, $f(1/2) = [0, 1/2]$, $f(t) = 1/2$ for $1/2 < t < 1$ and $f(1) = [1/2, 1]$. Then, $\dim(\varprojlim f) = 2$.*

The function in the following example includes flat spots but the inverse limit can be seen to be tree-like using Theorem 4.2. The example is a modification of

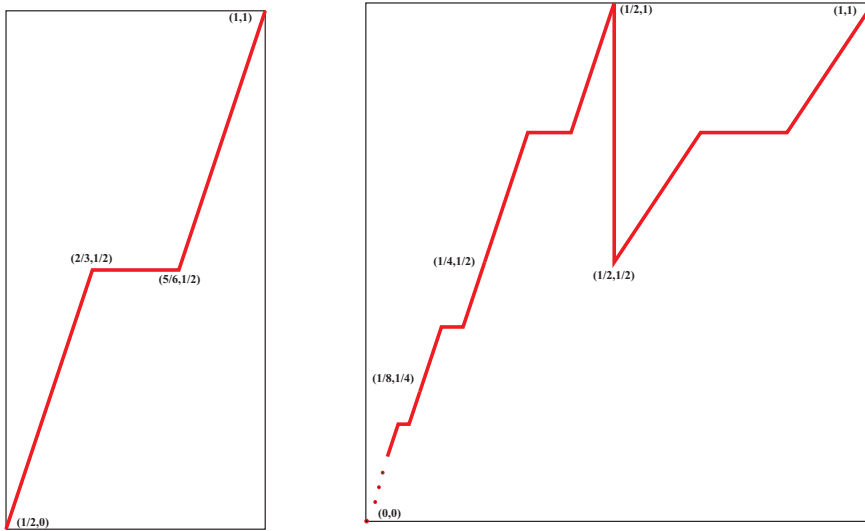


FIGURE 1. The graphs of the building block mapping φ (left) and the bonding function in Example 5.3

Example 3.5 of [4] which was studied further as Example 3.11 of [5]. Example 5.3 does not satisfy the conditions of Theorem 3.2 even if we change the value of $f(1/2)$ from $[1/2, 1]$ to $[0, 1]$.

Example 5.3. Let $\varphi : [1/2, 1] \rightarrow [0, 1]$ be the map consisting of three line intervals, one from $(1/2, 0)$ to $(2/3, 1/2)$ one from $(2/3, 1/2)$ to $(5/6, 1/2)$, and one from $(5/6, 1/2)$ to $(1, 1)$. Let $f : [0, 1] \rightarrow C([0, 1])$ be the upper semi-continuous function given by $f(0) = 0$; $f(1/2) = [1/2, 1]$; $f(t) = (1 + \varphi(t))/2$ for $1/2 < t \leq 1$; for $n \geq 1$, $f(t) = (1 + \varphi(2^n t))/2^n$ for $t \in (1/2^{n+1}, 1/2^n]$. Then, $\varprojlim \mathbf{f}$ is a tree-like continuum. (See Figure 1 for the graphs of φ and f .)

PROOF. Let $M = \varprojlim \mathbf{f}$; M is a continuum by Theorem 2.1. The set $Z = \{t \in [0, 1] \mid f(t) \text{ is nondegenerate}\} = \{1/2\}$ and $f^{-j}(Z) = \{1/2, 1/4, \dots, 1/2^{j+1}\}$ for each $j \in \mathbb{N}$, a totally disconnected set for each j . Theorem 4.2 yields that M is tree-like. \square

In our next example the inverse limit is an indecomposable tree-like continuum [5, Example 3.9] (original source is Example 3.4 of [4]). That it is tree-like is a consequence of Corollary 4.1.

Example 5.4. Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = 2t$ for $0 \leq t < 1/2$, $f(1/2) = [0, 1]$, and $f(t) = 2t - 1$ for $1/2 < t \leq 1$. Then, $\varprojlim \mathbf{f}$ is an indecomposable tree-like continuum.

In our next example the graph of the bonding function is homeomorphic to a $\sin(1/x)$ -curve. This example was first studied by Scott Varagona and he showed the inverse limit to be an indecomposable continuum [10, Theorem 3.2]. That it is tree-like is a consequence of Corollary 4.1.

Example 5.5. Let $f : [0, 1] \rightarrow C([0, 1])$ be the function whose graph is the union of straight line intervals joining $(1/2^n, 0)$ and $(1/2^{n-1}, 1)$ for all odd positive integers, straight line intervals joining $(1/2^{n-1}, 0)$ and $(1/2^n, 1)$ for all even positive integers, and the straight line interval joining $(0, 0)$ and $(0, 1)$ (a graph homeomorphic to a $\sin(1/x)$ -curve). Then, $\varprojlim \mathbf{f}$ is an indecomposable tree-like continuum.

Our final example is actually a sequence of examples. Each inverse limit is an infinite dimensional continuum and thus is not tree-like. However, as n increases, the difficulty of determining that the hypothesis of Theorem 4.2 is satisfied increases. The first term of the sequence is the function in Example 5.1. Recall from Section 2 the notation $G'_n = G'(f_1, f_2, \dots, f_n)$. We use the single bonding function version of this notation in the proof that the inverse limit in Example 5.6 is infinite dimensional. If $f : [0, 1] \rightarrow 2^{[0, 1]}$ and $f_i = f$ for each integer i , $1 \leq i \leq n$, then we denote $G'(f_1, f_2, \dots, f_n)$ by G'_n .

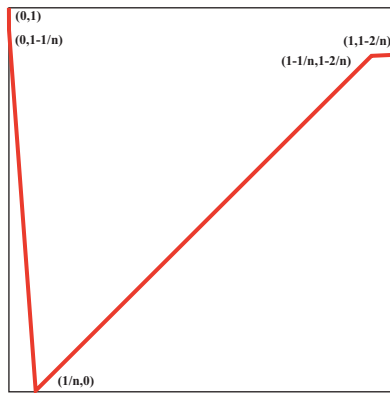


FIGURE 2. The graph of the bonding function in Example 5.6 with n approximately 16

Example 5.6. Let n be a positive integer. Let $f_n : [0, 1] \rightarrow C([0, 1])$ be the function whose graph consists of four straight line intervals, one from $(0, 1)$ to $(0, 1 - 1/n)$, one from $(0, 1 - 1/n)$ to $(1/n, 0)$, one from $(1/n, 0)$ to $(1 - 1/n, 1 - 2/n)$, and one from $(1 - 1/n, 1 - 2/n)$ to $(1, 1 - 2/n)$. Then, $\varprojlim \mathbf{f}_n$ is infinite dimensional. (See Figure 2 for a graph of f_n with n approximately 16).

PROOF. Let $n \in \mathbb{N}$ and $M = \varprojlim \mathbf{f}_n$. Then, G'_n contains $[1 - 1/n, 1] \times \{0\} \times \{1/n\} \times \cdots \times \{1 - 2/n\} \times [1 - 1/n, 1]$. It follows that M contains a 2-cell. In a similar manner we see that G'_{2n} contains a subset for which three factors that are the interval $[1 - 1/n, 1]$. In general, for each positive integer k , G'_{kn} contains a subset having $k + 1$ factors that are the interval $[1 - 1/n, 1]$ and all others are degenerate. It follows that M contains a cell of each finite dimension and is thus infinite dimensional. \square

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