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TREE-LIKENESS OF CERTAIN INVERSE LIMITS WITH SET-VALUED FUNCTIONS

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Abstract. Inverse limits with set-valued functions having graphs that are the union of mappings have attracted some attention over the past few years. In this paper we show that if the graph of a surjective set-valued function $f : [0, 1] \to 2^{[0,1]}$ is the union of two mappings having only one point (x, x) in common such that x is not the image of any other point of the interval under f, then its inverse limit is tree-like. Examples are included to show that various hypotheses in this theorem cannot be weakened.

1. Introduction

Interest in inverse limits with set-valued functions whose graphs are unions of mappings was initially kindled by potential applications to economics. However, it is a fascinating subject in its own right and has somewhat taken on a life of its own. In compiling a problem set for *An Introduction to Inverse Limits with Set-valued Functions* [4], it occurred to the author that it would be of interest to decide under what conditions an inverse limit with set-valued functions on $[0, 1]$ is a tree-like continuum; it is listed as Problem 6.49 in that book. In the literature on set-valued inverse limits, many examples that have been considered are tree-like while many others are not. In thinking some about this problem, the author decided to consider the case that the set-valued function has a graph that is the union of two mappings. The current literature concerned with inverse limits with set-valued functions having graphs that are the union of mappings includes [3] and [8] (see also [4, §2.7]). Here we show that under certain conditions a surjective set-valued function from $[0, 1]$ into

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$2^{[0,1]}$ having a graph that is the union of two mappings produces a tree-like continuum. We provide examples showing that without the conditions, the inverse limit may not be tree-like.

2. Definitions and Notation

We denote the collection of closed subsets of $[0,1]$ by $2^{[0,1]}$. A function $f : [0,1] \to 2^{[0,1]}$ is said to be *upper semi-continuous at the point x of* $[0,1]$ provided that if V is an open set that contains $f(x)$ then there is an open set U containing x such that if t is a point of U then $f(t) \subseteq V$. A function $f : [0,1] \to 2^{[0,1]}$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of $[0,1]$. If $f : [0,1] \to 2^{[0,1]}$, we say that f is *surjective* provided that for each $y \in [0,1]$ there is a point $x \in [0,1]$ such that $y \in f(x)$. If $f : [0,1] \to 2^{[0,1]}$ is a set-valued function, by the *graph* of f, denoted $G(f)$, we mean the subset of $[0,1] \times [0,1]$ that contains the point (x,y) if and only if $y \in f(x)$. It is known that if M is a subset of $[0,1] \times [0,1]$ such that $[0,1]$ is the projection of M to its set of first coordinates then M is closed if and only if M is the graph of a upper semi-continuous function [4, Theorem 1.2] (original source [5, Theorem 2.1]). In the case that f is upper semi-continuous and single-valued, i.e., $f(t)$ is degenerate for each $t \in [0,1]$, f is a continuous function. We call a continuous function a *mapping*; if $f : X \to Y$ is a surjective mapping, we denote this by $f : X \twoheadrightarrow Y$.

We denote by $\mathbb{N}$ the set of positive integers. If $\boldsymbol{s} = s_1, s_2, s_3, \dots$ is a sequence, we often denote the sequence in boldface type and its terms in italics. If $\boldsymbol{X}$ is a sequence such that $X_i = [0,1]$ for each $i \in \mathbb{N}$, we denote the product of terms of $\boldsymbol{X}$, $\prod_{i>0} X_i$, by $\mathcal{Q}$. The points of $\mathcal{Q}$ are sequences of numbers from $[0,1]$ so if $\boldsymbol{x} \in \mathcal{Q}$, it should not be a problem denoting $\boldsymbol{x}$ by $x_1, x_2, x_3, \dots$. However, we adopt the usual convention of enclosing the terms of $\boldsymbol{x}$ in parentheses, $\boldsymbol{x} = (x_1, x_2, x_3, \dots)$, to signify that $\boldsymbol{x}$ is a point of the product space. A metric d compatible with the product topology for $\mathcal{Q}$ is given by $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i>0} |x_i - y_i|/2^i$.

Suppose $\boldsymbol{X}$ is a sequence such that X_i is a closed subset of $[0,1]$ for each $i \in \mathbb{N}$ and $\boldsymbol{f}$ is a sequence of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ for each positive integer i. Such a pair of sequences $\{\boldsymbol{X}, \boldsymbol{f}\}$ is called an *inverse limit sequence*. By the *inverse limit* of the inverse limit sequence $\{\boldsymbol{X}, \boldsymbol{f}\}$, denoted $\varprojlim \boldsymbol{f}$, we mean the subset of $\mathcal{Q}$ that contains the point $(x_1, x_2, x_3, \dots)$ if and only if $x_i \in f_i(x_{i+1})$ for each positive integer i. In the case that each f_i is a mapping, the condition $x_i \in f_i(x_{i+1})$ becomes $x_i = f_i(x_{i+1})$ and the definition reduces to the definition of an inverse limit with mappings on a sequence of subintervals of $[0,1]$. In this paper, we make use of inverse limits with mappings to demonstrate certain properties of inverse limits with set-valued functions. For an inverse limit sequence $\{\boldsymbol{X}, \boldsymbol{f}\}$, the spaces X_i are called *factor spaces* and the functions f_i are called *bonding functions*. If X is a closed subset of $[0,1]$, $f : X \to 2^X$ is a set-valued function, and $\{\boldsymbol{X}, \boldsymbol{f}\}$ is an inverse limit sequence such that $X_i = X$ and $f_i = f$ for each $i \in \mathbb{N}$ (i.e., we have an inverse limit sequence with a single bonding function), we still denote the inverse limit by $\varprojlim \boldsymbol{f}$. We denote the projection from the inverse limit into the ith factor space by π_i. That these inverse limits are nonempty and compact is a consequence of [4, Theorem 1.6] or [5, Theorem 3.2]; they are metric spaces being subsets of the metric space $\mathcal{Q}$.

By a *continuum* we mean a compact, connected metric space. We use the term *dimension* in the standard sense as found in [2]. If M is a compactum, we use $\dim(M)$ to denote its dimension. A continuum is *tree-like* provided it is homeomorphic to an inverse limit on trees with mappings (or, equivalently, its dimension is 1 and every mapping of it to a finite graph is inessential).

3. Main Theorem

We now turn to the main theorem of this paper. Our proof relies on the following theorem of H. Cook. First we define some terms from Cook's paper. A collection $\mathcal{G}$ of continua is called a *clump* provided the union of all the elements of $\mathcal{G}$, denoted $\mathcal{G}^*$, is a continuum and there is a continuum C such that C is a proper subcontinuum of each element of $\mathcal{G}$ and C is the intersection of each two elements of $\mathcal{G}$. A clump is called *usc* (Cook uses the term "upper semi-continuous," but we use "usc" in this paper to draw a distinction between upper semi-continuous functions and upper semi-continuous clumps) provided that if $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ are two sequences of points of $\mathcal{G}^*$ converging to points p and q, respectively, of $\mathcal{G}^* - C$ and such that p_i and q_i belong to the same element of $\mathcal{G}$ for each $i \in \mathbb{N}$, then p and q belong to the same element of $\mathcal{G}$.

Theorem 3.1 ([1, Theorem 12])**.** *If $\mathcal{G}$ is a clump of tree-like continua such that $\mathcal{G}$ is usc and* $\dim(\mathcal{G}^*)$*=1, then $\mathcal{G}^*$ is tree-like.*

Our proof also uses the following theorems. They can be found in more general form in [4], the first being Theorem 2.11 for Theorem 3.2 (original source [8, Theorem 3.1]) and the second being Theorem 5.3 for Theorem 3.3 (original source [7, Theorem 5.3]).

Theorem 3.2. *Suppose $\mathcal{F}$ is a finite collection of mappings of* $[0,1]$ *into itself and f is the function whose graph is the union of the mappings in $\mathcal{F}$. If f is surjective and $G(f)$ is a continuum, then $\varprojlim \boldsymbol{f}$ is a continuum.*

Theorem 3.3. *Suppose $\mathcal{F}$ is a finite collection of mappings of $[0,1]$ into itself and f is the set-valued function whose graph is the union of the mappings in $\mathcal{F}$. Because $\dim(f(t)) = 0$ for each $t \in [0,1]$, $\dim(\varprojlim \boldsymbol{f}) \le 1$.*

Theorem 3.4. *Suppose f_1 and f_2 are mappings of $[0,1]$ into $[0,1]$ such that the only point of intersection of f_1 and f_2 is a common fixed point x such that $f_1^{-1}(x) = f_2^{-1}(x) = \{x\}$. If $f : [0,1] \to 2^{[0,1]}$ is the upper semi-continuous function whose graph is the set-theoretic union of f_1 and f_2 and f is surjective, then $\varprojlim \boldsymbol{f}$ is a tree-like continuum.*

Proof. Let $M = \varprojlim \boldsymbol{f}$; M is a continuum by Theorem 3.2. Because $f(x) = \{x\}$ and $f(t)$ contains only two points for $t \neq x$, $\dim(f(t))$=0 for each $t \in [0,1]$. By Theorem 3.3, $\dim(M) \le 1$. Because f is surjective, M is nondegenerate, so $\dim(M)$=1. From the hypothesis that $f_1(x) = f_2(x) = x$ and $f_1^{-1}(x) = f_2^{-1}(x) = \{x\}$, it follows that

$(*)$ if $\boldsymbol{p} \in M$ and $p_i = x$ for some $i \in \mathbb{N}$, then $\boldsymbol{p} = (x,x,x,\dots)$.

The remainder of the proof consists of identifying a usc clump $\mathcal{G}$ such that $M = \mathcal{G}^*$ so that we may apply Theorem 3.1. Let $\mathcal{G} = \{H \subseteq M \mid H$ is a subcontinuum of M and there is a sequence $\boldsymbol{g}$ of mappings of $[0,1]$ into itself such that $g_i \in \{f_1, f_2\}$ for each $i \in \mathbb{N}$ and $H = \varprojlim \boldsymbol{g}\}$. It is clear that $M = \mathcal{G}^*$. Each element of $\mathcal{G}$ is an inverse limit with mappings on $[0,1]$ and thus is tree-like. To see that $\mathcal{G}$ is a clump, we first observe that if H and K belong to $\mathcal{G}$ and $H \neq K$, then $H \cap K = \{(x,x,x,\dots)\}$. Indeed, suppose $\boldsymbol{y} \in H \cap K$ with H and K in $\mathcal{G}$. There exist sequences $\boldsymbol{h}$ and $\boldsymbol{k}$ of mappings such that $h_i, k_i \in \{f_1, f_2\}$ for each $i \in \mathbb{N}$, $H = \varprojlim \boldsymbol{h}$, and $K = \varprojlim \boldsymbol{k}$. If $H \neq K$, then there is a positive integer i such that $h_i \neq k_i$. Thus, $y_i = f_1(y_{i+1}) = f_2(y_{i+1})$, and therefore $y_i = y_{i+1} = x$. Then, by $(*)$, $\boldsymbol{y} = (x,x,x,\dots)$.

To see that $\mathcal{G}$ is usc, suppose $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}, \dots$ and $\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3}, \dots$ are two sequences of points of $\mathcal{G}^*$ such that $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}, \dots$ converges to $\boldsymbol{p} \neq (x,x,x,\dots)$, $\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3}, \dots$ converges to $\boldsymbol{q} \neq (x,x,x,\dots)$, and $\boldsymbol{p_i}$ and $\boldsymbol{q_i}$ belong to the same element of $\mathcal{G}$ for each $i \in \mathbb{N}$. For each positive integer i, there is a sequence $g_1^i, g_2^i, g_3^i, \dots$ such that $g_k^i \in \{f_1, f_2\}$ for each $k \in \mathbb{N}$ and $\boldsymbol{p_i}, \boldsymbol{q_i} \in \varprojlim \boldsymbol{g^i}$. Assume $\boldsymbol{p} \in \varprojlim \boldsymbol{a}$ and $\boldsymbol{q} \in \varprojlim \boldsymbol{b}$ with $a_i, b_i \in \{f_1, f_2\}$ for each $i \in \mathbb{N}$. If $\boldsymbol{p}$ and $\boldsymbol{q}$ do not belong to the same element of $\mathcal{G}$, $\boldsymbol{q} \notin \varprojlim \boldsymbol{a}$. Thus, there is a positive integer j such that $a_j(\pi_{j+1}(\boldsymbol{q})) \neq \pi_j(\boldsymbol{q})$. Assume $a_j = f_1$ (the case that $a_j = f_2$ is similar and is omitted). Then, $f_1(\pi_{j+1}(\boldsymbol{p})) = \pi_j(\boldsymbol{p})$ and $f_1(\pi_{j+1}(\boldsymbol{q})) \neq \pi_j(\boldsymbol{q})$ while $f_2(\pi_{j+1}(\boldsymbol{q})) = \pi_j(\boldsymbol{q})$. Consider the sequence $g_j^1, g_j^2, g_j^3, \dots$ of mappings. Because $\boldsymbol{p_i}$ and $\boldsymbol{q_i}$ belong to $\varprojlim \boldsymbol{g^i}$ for each $i \in \mathbb{N}$, it follows that $g_j^i(\pi_{j+1}(\boldsymbol{q_i})) = \pi_j(\boldsymbol{q_i})$ and $g_j^i(\pi_{j+1}(\boldsymbol{p_i})) = \pi_j(\boldsymbol{p_i})$ for each positive integer i. There are two possibilities: (1) there is a positive integer N

such that $g^i_j = f_1$ for $i \geq N$ and (2) there is an increasing sequence $n_1, n_2, n_3, \ldots$ such that $g^{n_i}_j = f_2$ for each positive integer i. Suppose (1) is true. Then, for $i \geq N$, $f_1(\pi_{j+1}(\boldsymbol{q_i})) = \pi_j(\boldsymbol{q_i})$. Because $\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3}, \ldots$ converges to $\boldsymbol{q}$, it follows that $f_1(\pi_{j+1}(\boldsymbol{q})) = \pi_j(\boldsymbol{q})$, contradicting that $a_j = f_1$ and $a_j(\pi_{j+1}(\boldsymbol{q})) \neq \pi_j(\boldsymbol{q})$. Suppose (2) holds. Then, for each $i \in \mathbb{N}$, $f_2(\pi_{j+1}(\boldsymbol{p_{n_i}})) = \pi_j(\boldsymbol{p_{n_i}})$. Because $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}, \ldots$ converges to $\boldsymbol{p}$, it follows that $f_2(\pi_{j+1}(\boldsymbol{p})) = \pi_j(\boldsymbol{p})$. Because $a_j = f_1$, it is also true that $f_1(\pi_{j+1}(\boldsymbol{p})) = \pi_j(\boldsymbol{p})$. Thus, $\pi_{j+1}(\boldsymbol{p}) = x$, also a contradiction by $(*)$. So $\boldsymbol{p}$ and $\boldsymbol{q}$ belong to the same element of $\mathcal{G}$ and we have that $\mathcal{G}$ is usc. By Theorem 3.1, M is tree-like. □

4. Examples

That the two mappings in Theorem 3.4 cannot have two points of intersection can be seen from the following example from [4, Example 2.11] (original source [5, Example 4]).

Example 4.1. *Let $f : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function given by $f(t) = \{t + 1/2, 1/2 - t\}$ for $0 \leq t \leq 1/2$ and $f(t) = \{3/2 - t, t - 1/2\}$ for $1/2 < t \leq 1$. Then $G(f)$ is the union of two mappings having $(0, 1/2)$ and $(1, 1/2)$ in common, but $\varprojlim \boldsymbol{f}$ contains a simple closed curve and so is not tree-like. (See Figure 1 for the graph of f.)*

Proof. Let $M = \varprojlim \boldsymbol{f}$. There are numerous simple closed curves in M. We exhibit one as follows. Let $J_1 = [0, 1/2]$ and $J_2 = [1/2, 1]$. Let $f_1 : J_1 \twoheadrightarrow J_1$ be given by $f_1(t) = 1/2 - t$, $f_2 : J_1 \twoheadrightarrow J_2$ be given by $f_2(t) = 1/2 + t$, $f_3 : J_2 \twoheadrightarrow J_1$ be given by $f_3(t) = t - 1/2$, and $f_4 : J_2 \twoheadrightarrow J_2$ be given by $f_4(t) = 3/2 - t$. Let $\boldsymbol{a}$ be the sequence every term of which is f_1 and $\boldsymbol{d}$ be the sequence every term of which is f_4. Let $\boldsymbol{b}$ be the sequence having all odd numbered terms f_2 and all even numbered terms f_3; let $\boldsymbol{c}$ be the sequence having all odd numbered terms f_3 and all even numbered terms f_2. Let $\alpha = \varprojlim \boldsymbol{a}$, $\beta = \varprojlim \boldsymbol{b}$, $\gamma = \varprojlim \boldsymbol{c}$, and $\delta = \varprojlim \boldsymbol{d}$. Each of these four inverse limits is an arc being the inverse limit on intervals with homeomorphisms [6, Theorem 200]. Further, α has endpoints $(0, 1/2, 0, 1/2, \ldots)$ and $(1/2, 0, 1/2, 0, \ldots)$; β has endpoints $(1/2, 0, 1/2, 0, \ldots)$ and $(1, 1/2, 1, 1/2, \ldots)$; γ has endpoints $(0, 1/2, 0, 1/2, \ldots)$ and $(1/2, 1, 1/2, 1, \ldots)$; the endpoints of δ are $(1, 1/2, 1, 1/2, \ldots)$ and $(1/2, 1, 1/2, 1, \ldots)$. It is not difficult to verify that each two of these four arcs intersect only at one common endpoint. For instance, if $\boldsymbol{x} \in \alpha \cap \beta$ then $f_1(x_2) = f_2(x_2)$; thus $x_2 = 0$ and $x_1 = 1/2$. However, $f_1(x_3) = f_3(x_3)$ so $x_3 = 1/2$. Continuing, we see that $\boldsymbol{x} = (1/2, 0, 1/2, 0, \ldots)$. It follows that the union of the four arcs is a simple closed curve. □

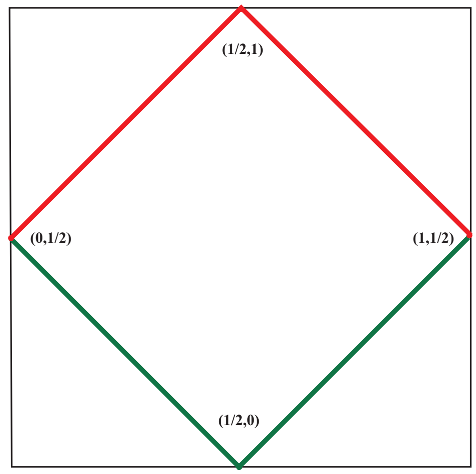

FIGURE 1. The graph of the function in Example 4.1.

That the two mappings in Theorem 3.4 must intersect at a common fixed point may be seen from the following example.

Example 4.2. *Let $f_1 : [0,1] \twoheadrightarrow [0,1]$ be the piecewise linear mapping whose graph consists of two straight line intervals in $[0,1]^2$, one from $(0,0)$ to $(1/2,3/4)$ and the other from $(1/2,3/4)$ to $(1,1)$. Let $f_2 : [0,1] \twoheadrightarrow [0,1]$ be the piecewise linear mapping whose graph consists of two straight line intervals in $[0,1]^2$, one from $(0,1)$ to $(1/2,3/4)$ and the other from $(1/2,3/4)$ to $(1,0)$. If $f : [0,1] \to 2^{[0,1]}$ is the upper semi-continuous function such that $G(f) = f_1 \cup f_2$, then $\varprojlim \boldsymbol{f}$ contains a simple closed curve and so is not tree-like. (See Figure 2 for the graph of f.)*

Proof. Let $M = \varprojlim \boldsymbol{f}$. For the reader's convenience we note that $f(t) = \{3t/2, 1 - t/2\}$ for $0 \le t \le 1/2$ and $f(t) = \{(t+1)/2, -3(t-1)/2\}$ for $1/2 < t \le 1$. To see that M contains a simple closed curve, we first identify four arcs lying in M. Let $\boldsymbol{a}$ be the sequence every term of which is the mapping f_1 and let $A_1 = \varprojlim \boldsymbol{a}$. Let $\boldsymbol{b}$ be the sequence such that $b_1 = f_1, b_2 = f_2$, and $b_i = f_1$ for $i \ge 3$; let $A_2 = \varprojlim \boldsymbol{b}$. Let $\boldsymbol{c}$ be the sequence such that $c_1 = f_2$ and $c_i = f_1$ for $i \ge 2$; let $A_3 = \varprojlim \boldsymbol{c}$. Finally, let $\boldsymbol{d}$ be the sequence such that $d_1 = d_2 = f_2$ and $d_i = f_1$ for $i \ge 3$; let $A_4 = \varprojlim \boldsymbol{d}$. That A_i is an arc for $i \in \{1,2,3,4\}$ is a consequence of the fact that f_1 and f_2 are homeomorphisms [6, Theorem 18]. The only point common to A_1 and A_2 is the point $(7/8, 3/4, 1/2, 1/3, 2/9, \dots)$ for if $\boldsymbol{x} \in A_1 \cap A_2$ and $x_3 \ne 1/2$ then $f_1(x_3) \ne f_2(x_3)$. In a similar manner we may show that $A_1 \cap A_3 = \{(3/4, 1/2, 1/3, 2/9, \dots)\}$, $A_2 \cap A_4 = \{(3/4, 1/2, 2/3, 4/9, \dots)\}$, and $A_3 \cap A_4 = \{(3/8, 3/4, 1/2, 1/3, 2/9, \dots)\}$. That A_1 and A_4 do not intersect may be seen as follows. If $\boldsymbol{x} \in A_1$ and $f_2(x_2) = x_1$ then $x_1 = 3/4$ and $x_2 = 1/2$. Thus, $x_3 = 1/3$ but $f_2(1/3) \neq 1/2$ so $\boldsymbol{x} \notin A_4$. Similarly, $A_2 \cap A_3 = \emptyset$. It follows that $A_1 \cup A_2 \cup A_3 \cup A_4$ contains a simple closed curve and M is not tree-like. □

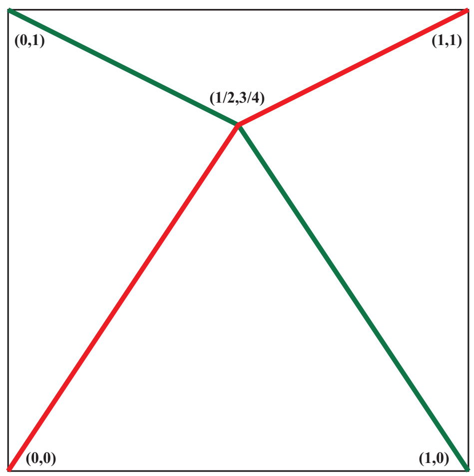

FIGURE 2. The graph of the function in Example 4.2.

That the condition that $f_1^{-1}(x) = f_2^{-1}(x) = \{x\}$ in Theorem 3.4 is needed may be seen from the following example.

Example 4.3. *Let f_1 be the identity, Id, on $[0,1]$ and f_2 be the map given by $f_2(t) = 2t + 1/2$ for $0 \leq t \leq 1/4$, $f_2(t) = -2t + 3/2$ for $1/4 < t \leq 1/2$, and $f_2(t) = 1 - t$ for $1/2 < t \leq 1$. If f is the function whose graph is the union of f_1 and f_2, then $f^{-1}(1/2) \neq \{1/2\}$ and $\varprojlim \boldsymbol{f}$ is not tree-like. (See Figure 3 for the graph of f.)*

Proof. Let $M = \varprojlim \boldsymbol{f}$. We show that M contains two subcontinua whose intersection is not connected from which it follows that M is not tree-like. Let $\boldsymbol{h}$ be the sequence the first two terms of which are f_2 and all other terms are f_1; let $H = \varprojlim \boldsymbol{h}$. Let $\boldsymbol{k}$ be the sequence the second term of which is f_2 and all other terms are f_1; let $K = \varprojlim \boldsymbol{k}$. The points $\boldsymbol{p} = (1/2, 1/2, 1/2, \dots)$ and $\boldsymbol{q} = (1/2, 1/2, 0, 0, 0, \dots)$ are points of $H \cap K$. Suppose $\boldsymbol{x} \in H \cap K$. If $x_1 \in [0, 1/2]$, because $f_2(x_2) = x_1$, we see that $x_2 \in [1/2, 1] \cup \{0\}$. However, $f_1(0) = 0$ and $f_2(0) = 1/2$, so $x_2 \neq 0$. Because $f_1(x_2) = x_1$, we note that $x_2 \in [0, 1/2]$. Thus, $x_2 = 1/2$, and so $x_1 = 1/2$. Because $x_2 = 1/2$, $x_3 \in \{0, 1/2\}$. From $f_j = Id$ for $j \geq 3$, it follows that $\boldsymbol{x} \in \{\boldsymbol{p}, \boldsymbol{q}\}$. Similarly, if $x_1 \in [1/2, 1]$, $\boldsymbol{x} \in \{\boldsymbol{p}, \boldsymbol{q}\}$. Therefore, $H \cap K = \{\boldsymbol{p}, \boldsymbol{q}\}$. □

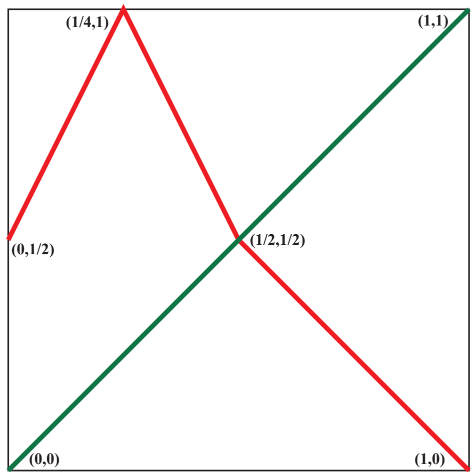

Figure 3. The graph of the function in Example 4.3.

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