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INVERSE LIMITS OF UPPER SEMI-CONTINUOUS FUNCTIONS THAT ARE UNIONS OF MAPPINGS

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ABSTRACT. In this paper we study inverse limits with upper semi-continuous functions that are unions of mappings. We show that if f is an upper semi-continuous function such that f is the union of countably many mappings of a continuum X into itself, one of which is universal with respect to the countable collection of mappings, then $\varprojlim \boldsymbol{f}$ is a continuum. Moreover, the dimension of $\varprojlim \boldsymbol{f}$ is not greater than m if X is m-dimensional for some positive integer m. Under certain conditions when $X = [0,1]$, the hypothesis of universality can be relaxed while producing a similar result for unions of maps that are not necessarily surjective.

1. Introduction

In [5], William S. Mahavier introduced the study of inverse limits using upper semi-continuous bonding functions where he looked at inverse limits with closed subsets of $[0,1] \times [0,1]$. Later, he and this author extended the definition to inverse limits of inverse sequences of compact Hausdorff spaces with upper semi-continuous bonding functions [4]. In this note, we study inverse limits using an interesting class of upper semi-continuous functions as bonding functions. These are functions that consist of the set theoretic union of a collection of mappings. Specifically, we show that if X is a continuum and $f : X \to 2^X$ is an upper semi-continuous function

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from X into the closed subsets of X such that f is the union of a collection of mappings of X into itself, one of which is universal with respect to the collection whose union is f, then the inverse limit is a continuum. Moreover, if m is a positive integer such that X is m-dimensional and the collection of mappings is countable, the dimension of the inverse limit is not greater than m. Conditions are also given in the specific case when $X = [0, 1]$ that insure that the inverse limit is a one-dimensional continuum (see Theorem 4.2). Without some conditions on the mappings, the statement is false since the union of the mapping that is identically 0 on $[0, 1]$ with the mapping that is identically 1 on $[0, 1]$ yields a Cantor set for its inverse limit; see Figure 1. Although the subject of this article is of interest in and of itself, some who are looking at applications of inverse limits in economics have asked about the nature of such inverse limits.

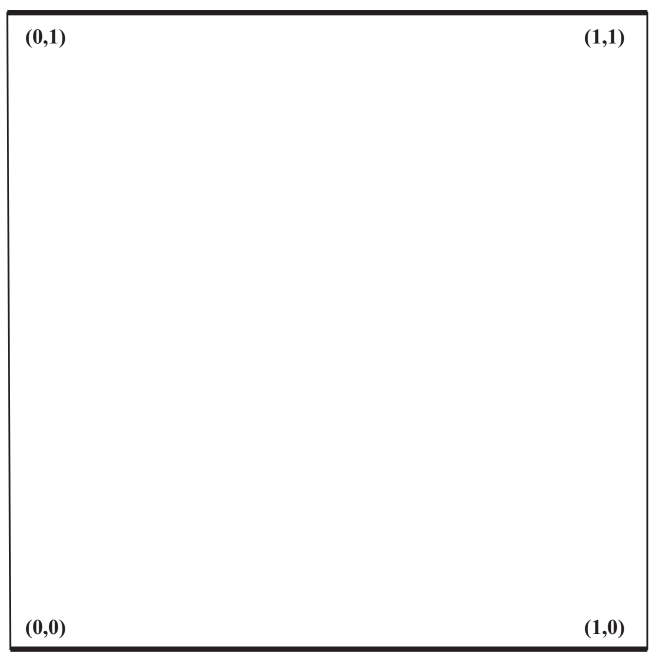

FIGURE 1. An upper semi-continuous function that is the union of two mappings having a Cantor set as its inverse limit

By a *continuum* we mean a compact, connected subset of a metric space. Suppose X and Y are continua and $\mathcal{F}$ is a collection of mappings of X into Y. A mapping $f : X \to Y$ is said to be *universal with respect to* $\mathcal{F}$ provided that if $g \in \mathcal{F}$, then there is a point $x \in X$ such that $f(x) = g(x)$, (i.e., f has a *coincidence point* with each member of $\mathcal{F}$).

If $f_1, f_2, f_3, \dots$ is a sequence of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$, by the *inverse limit* of the sequence $\boldsymbol{f} = f_1, f_2, f_3, \dots$, denoted $\varprojlim \boldsymbol{f}$, is meant the subset of $\prod_{i>0} X_i$ that contains the point $\boldsymbol{x} = (x_1, x_2, x_3, \dots)$ if and only if x_i belongs to $f_i(x_{i+1})$ for each i. The reader should note that we use the usual metric on the product space; i.e., $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i>0} 2^{-i} d_i(x_i, y_i)$ where d_i is a metric for X_i that is bounded by 1. For numerous examples and basic results, such as some sufficient conditions that the inverse limit be a continuum when each factor space is a continuum, see [5] and [4]. The reader should note that we adopt from those articles the convention of denoting a sequence in boldface type and the terms of the sequence in italic type.

2. Dimension

If G is a finite collection of sets and m is a positive integer, we say that the *order* of G is m provided m is the largest of the integers i such that there are $i+1$ members of G with a common element. If G and H are collections of sets, we say that H *refines* G provided that for each element h of H, there is an element g of G such that $h \subset g$. If m is a positive integer, the compact metric space X is said to have *dimension not greater than* m, written $\dim(X) \le m$, provided, for each positive number ε, there is a collection of open sets covering X that has mesh less than ε (i.e., the largest of the diameters of the sets in the collection is less than ε) and order not greater than m. We say the dimension of X is m, written $\dim(X) = m$, provided $\dim(X) \le m$ and $\dim(X) \not\le m-1$. It is convenient to use this definition of dimension (sometimes called *covering dimension*) in the study of inverse limits. For compact metric spaces it is equivalent to the usual definition of small inductive dimension [3, Theorem V 8, p. 67].

We now turn to showing that, in the case that the factor spaces are continua each of dimension not greater than m and the bonding functions are upper semi-continuous functions each of which is a union of countably many mappings, the dimension of the inverse limit is not greater than m. A slightly more general version of the first part of Theorem 2.1 is found in [8, Theorem 5.3]. However, our proof is not long and we include it for the sake of completeness. In [4], it is shown in the proofs of Theorem 3.1 and Theorem 3.2 that if $X_1, X_2, X_3, \dots$ is a sequence of compact Hausdorff spaces and $f_1, f_2, f_3, \dots$ is a sequence of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ for $i = 1, 2, 3, \dots$, then $\varprojlim \boldsymbol{f} = \bigcap_{i>2} G_n$ where G_n is the compact set, $\{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i < n\}$. Although this is not difficult to show, we will not reprove it here, but we do make use of this in the proof of the next theorem.

Theorem 2.1. *Suppose $X_1, X_2, X_3, \dots$ is a sequence of continua, m is a positive integer, and $f_1, f_2, f_3, \dots$ is a sequence of upper semi-continuous functions such that $f_i : X_{i+1} \to X_i$ and X_i is a continuum of dimension not greater than m for each positive integer i. If f_i is the union of countably many mappings $f_1^i, f_2^i, f_3^i, \dots$ of X_{i+1} into X_i for each i , then the dimension of $\varprojlim \boldsymbol{f}$ is not greater than m. Moreover, if $m = 1$ and there is a sequence $\boldsymbol{g}$ such that $g_i \in \{f_1^i, f_2^i, f_3^i \dots\}$ for each i and $\varprojlim \boldsymbol{g}$ is non-degenerate, then the dimension of $\varprojlim \boldsymbol{f}$ is 1.*

Proof: As we noted in the paragraph just preceding the statement of this theorem, $\varprojlim \boldsymbol{f} = \bigcap_{n>2} G_n$ where $G_n = \{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } i < n\}$. For each integer $n > 1$, let $G'_n = \{\boldsymbol{x} \in \prod_{i=1}^n X_i \mid$ there exists a finite sequence $g_1,\ g_2,\ \dots,\ g_{n-1}$ with $g_j \in \{f_1^j, f_2^j, f_{3_i}^j, \dots\}$ and $x_j = g_j(x_{j+1})$ for $1 \le j < n\}$. Note that $G_n = G'_n \times \prod_{i>n} X_i$ and G'_n is compact.

If $g_1, g_2, \dots, g_{n-1}$ is a finite sequence of maps such that $g_j \in \{f_1^j,\ f_2^j,\ f_{3_j}^j,\ \dots\}$ for $1 \le j < n$, then X_n is homeomorphic to $\Gamma(g_1, g_2, \dots, g_{n-1}) = \{\boldsymbol{x} \in G'_n \mid x_i = g_i(x_{i+1}) \text{ for } 1 \le i < n\}$ (the map that takes the point t of X_n to the point $(g_1 \circ g_2 \circ \cdots \circ g_{n-1}(t),\ g_2 \circ \cdots \circ g_{n-1}(t),\ \cdots,\ g_{n-1}(t), t)$ of $\Gamma(g_1, g_2, \dots, g_{n-1})$ is a homeomorphism). Thus, the dimension of $\Gamma(g_1, g_2, \dots, g_{n-1})$ is not greater than m. It follows that the dimension of G'_n is not greater than m since G'_n is the union of the countably many members of $\{\Gamma(g_1, g_2, \dots, g_{n-1}) \mid g_i \in \{f_1^i,\ f_2^i, f_{3_i}^i,\ \dots\} \text{ for } 1 \le i < n\}$; see [7, Theorem 7.1, p. 33] or [3, Theorem III 2, p. 30].

Suppose $n > 2$. There is a finite collection $\mathcal{H}'_n$ of open subsets of $\prod_{i=1}^n X_i$ covering G'_n such that the order of $\mathcal{H}'_n$ is not greater than m and the mesh of $\mathcal{H}'_n$ is less than $1/2^n$. Note that if $R \in \mathcal{H}'_n$, then $R \times \prod_{i>n} X_i$ is a subset of $\prod_{i>0} X_i$ having diameter less than $1/2^{n-1}$. Let $\mathcal{H}_n = \{R \times \prod_{i>n} X_i \mid R \in \mathcal{H}'\}$. Then $\mathcal{H}_n$ is a finite collection of open sets covering G_n (and therefore $\varprojlim \boldsymbol{f}$) such that the order of $\mathcal{H}_n$ is not greater than m and the mesh of $\mathcal{H}_n$ is less than $1/2^{n-1}$.

For each positive integer n, there is a finite open cover of $\varprojlim \boldsymbol{f}$ of order not greater than m and mesh less than $1/2^{n-1}$; therefore, the dimension of $\varprojlim \boldsymbol{f}$ is not greater than m.

Moreover, if $m = 1$ and there is a sequence $\boldsymbol{g}$ such that $g_i \in \{f^i_1, f^i_2, f^i_{3_i}\}, \ldots$ for each positive integer i and $\varprojlim \boldsymbol{g}$ is non-degenerate, then $\varprojlim \boldsymbol{f}$ contains a non-degenerate continuum so its dimension is 1. □

3. Unions of mappings of continua

Lemma 3.1. *Suppose X is a continuum and $f_i : X \to X$ is a mapping of X into itself for each positive integer i, $g : X \to X$ is a mapping, and n is a positive integer such that f_n and g have a coincidence point. If f_i is surjective for $i > n$ and $\boldsymbol{\varphi}$ is a sequence of mappings such that $\varphi_i = f_i$ for $i \neq n$ and $\varphi_n = g$, then $\varprojlim \boldsymbol{f}$ and $\varprojlim \boldsymbol{\varphi}$ have a point in common.*

Proof: There is a point t of X such that $f_n(t) = g(t)$. Since f_i is surjective for each $i > n$, there is a point $\boldsymbol{x}$ of $\varprojlim \boldsymbol{f}$ such that $x_{n+1} = t$. Since $g(t) = f_n(t)$, $\boldsymbol{x}$ is in $\varprojlim \boldsymbol{\varphi}$. □

Theorem 3.2. *If $\mathcal{F}$ is a collection of mappings of a non-degenerate continuum X into itself one of which is surjective and universal with respect to $\mathcal{F}$ and f is a closed subset of $X \times X$ that is the set theoretic union of the collection $\mathcal{F}$, then $f : X \to 2^X$ is an upper semi-continuous function and $\varprojlim \boldsymbol{f}$ is a continuum.*

Proof: Since f is a closed subset of $X \times X$ and each point of X is a first coordinate of some point of f, f is upper semi-continuous, [4, Theorem 2.1, p. 120]. Since $\varprojlim \boldsymbol{f}$ is compact, we need only to show that this inverse limit is connected. Suppose f_1 is a member of $\mathcal{F}$ that is surjective and universal with respect to $\mathcal{F}$. Since f_1 is surjective, the ordinary inverse limit, $\varprojlim \boldsymbol{f_1}$, is a non-degenerate continuum. Choose a point $\boldsymbol{x} \in \varprojlim \boldsymbol{f_1}$ and let $\boldsymbol{y}$ be a point of $\varprojlim \boldsymbol{f}$. There exists a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ such that $\varphi_i \in \mathcal{F}$ and $\varphi_i(y_{i+1}) = y_i$ for each positive integer i. Let $C_1 = \varprojlim \boldsymbol{f}_1$, and, if n is a positive integer with $n > 1$, let C_n be the inverse limit of the sequence $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}, f_1, f_1, f_1, \ldots$. For each n, C_n is a continuum (in fact, C_n is homeomorphic to $\varprojlim \boldsymbol{f}_1$) and, by Lemma 3.1, $C_n \cap C_{n+1} \neq \emptyset$. Thus, $\bigcup_{i>0} C_i$ is connected. Moreover, for each n, since f_1 is surjective, there is a point $\boldsymbol{p^n}$ of C_n such that $\pi_i(\boldsymbol{p^n}) = y_i$ for $i \leq n$. It follows that $\boldsymbol{y} \in \overline{C}$. Since $\varprojlim \boldsymbol{f}$ is the union of a collection of continua all containing $\boldsymbol{x}$, $\varprojlim \boldsymbol{f}$ is connected. □

Since the set theoretic union of a finite collection of mappings of a continuum X into itself is closed, the theorem below follows from Theorem 3.2.

Theorem 3.3. *If $\mathcal{F}$ is a finite collection of mappings of a continuum into itself, one of which is surjective and universal with respect to $\mathcal{F}$ and f is the set theoretic union of the maps in $\mathcal{F}$, then $\varprojlim \boldsymbol{f}$ is a continuum.*

The inverse limit of the upper semi-continuous function depicted in Figure 2 can be seen to be a continuum using Theorem 3.3. Its inverse limit is known to be a simple fan.

4. Unions of mappings of $[0,1]$

We end this article with some results on inverse limits on $[0,1]$ with upper semi-continuous functions that are unions of finitely many mappings. The reader should note that Theorem 4.2 generalizes Theorem 3.3 as applied to upper semi-continuous functions on $[0,1]$. In Theorem 4.2, we do not assume surjectivity of any of the maps in $\mathcal{F}$. In Figure 3, we depict an upper semi-continuous function that is the union of two maps of $[0,1]$ into itself neither of which is surjective. If one takes the upper map to be f_1 and $p_g = 1$ (see the statement of Theorem 4.2), the collection consisting of the two maps shown satisfies the hypothesis of Theorem 4.2, so its inverse limit is a one-dimensional continuum. This inverse limit is known to be the Hurewicz continuum from [2].

In the proof of Lemma 3.1, as well as in the proof of Theorem 3.2, we made use of the well-known fact that if $f : X \twoheadrightarrow X$ is a

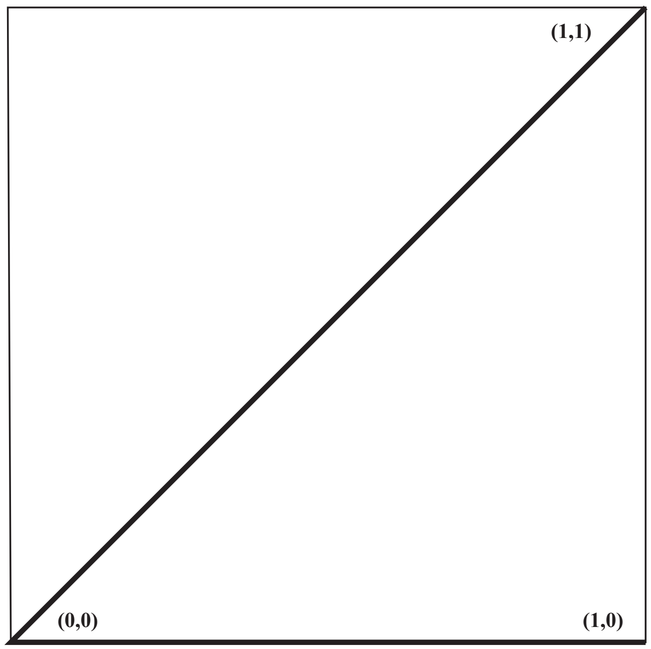

FIGURE 2. An upper semi-continuous function that is the union of two mappings having a simple fan as its inverse limit

surjective map of a compact space onto itself, then $\varprojlim \boldsymbol{f}$ is non-empty. The following lemma is a slight generalization of this fact.

Lemma 4.1. *If $f : X \to X$ is a mapping of a compact metric space X into itself such that $f(f(X)) = f(X)$ and t is a point of $f(X)$, then there is a point $\boldsymbol{x}$ of $\varprojlim f$ such that $x_1 = t$.*

Proof: Let t be a point of $f(X)$ and let $x_1 = t$. Since $f(f(X)) = f(X)$, there is a point x_2 of $f(X)$ such that $f(x_2) = x_1$. Similarly, since x_2 is in $f(X) = f(f(X))$, there is a point x_3 of $f(X)$ such that $f(x_3) = x_2$. Continuing in this manner, we obtain a point $\boldsymbol{x}$ of $\varprojlim \boldsymbol{f}$ such that $x_1 = t$. □

Theorem 4.2. *Suppose $\mathcal{F}$ is a collection of mappings of $[0, 1]$ into itself such that the union f of the collection $\mathcal{F}$ is closed. Suppose further that $\mathcal{F}$ contains a mapping f_1 with the following properties:*

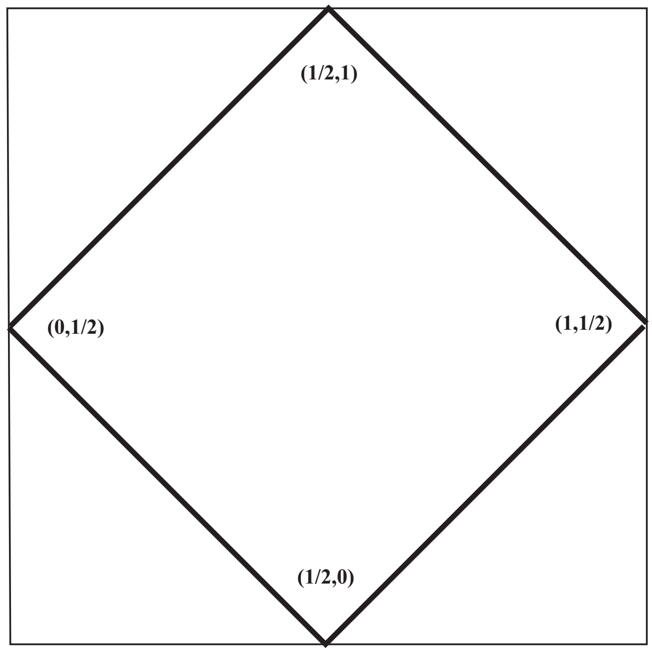

FIGURE 3. An upper semi-continuous function that is the union of two non-surjective mappings that has the Hurewicz continuum as its inverse limit

(1) *$f_1([0,1])$ is non-degenerate;*
(2) *if $g \in \mathcal{F}$ there is a point p_g of $f_1([0,1])$ such that $f_1(p_g) = g(p_g)$; and*
(3) *if $g \in \mathcal{F}$, then $g(f_1([0,1])) = g([0,1])$.*

If f is the set theoretic union of all the elements of $\mathcal{F}$, then $\varprojlim \boldsymbol{f}$ is a one-dimensional continuum.

Proof: Observe that since f is closed, it is upper semi-continuous.

Choose a point $\boldsymbol{y}$ in $\varprojlim \boldsymbol{f}$. There exists a sequence $\varphi_1, \varphi_2, \varphi_3, \dots$ such that φ_i is in $\mathcal{F}$ and $\varphi_i(y_{i+1}) = y_i$ for each positive integer i. Let C_1 be the inverse limit of the sequence $f_1, f_1, f_1, \dots$, and if $n > 1$, let C_n be the inverse limit of the sequence $\varphi_1, \varphi_2, \dots, \varphi_{n-1}, f_1, f_1, \dots$. Using condition (2), it follows from Lemma 3.1 that C_i and C_{i+1} have a point in common for each positive integer i. Thus, $C_1 \cup C_2 \cup C_3 \cup \dots$ is connected. Using Lemma 4.1, we obtain a point $\boldsymbol{x}$

of $\varprojlim \boldsymbol{f_1}$ such that $x_1 = y_n$. The point $\boldsymbol{p_n} = (y_1, y_2, \ldots, y_n, x_2, x_3, \ldots)$ belongs to C_n and the distance from $\boldsymbol{y}$ to $\boldsymbol{p_n}$ is less than $1/2^n$. Thus, $\boldsymbol{y}$ belongs to $\mathrm{cl}(C_1 \cup C_2 \cup C_3 \cup \ldots)$. Since each point of $\varprojlim \boldsymbol{f}$ belongs to a continuum lying in $\varprojlim \boldsymbol{f}$ that contains the continuum $\varprojlim \boldsymbol{f_1}$, $\varprojlim \boldsymbol{f}$ is a continuum.

From condition (1) and Lemma 4.1, it follows that $\varprojlim \boldsymbol{f_1}$ is non-degenerate. Thus, the dimension of $\varprojlim \boldsymbol{f}$ is one by Theorem 2.1. □

Any mapping $f : I \twoheadrightarrow I$ of $I = [0,1]$ onto itself is universal. Therefore, if a finite family $\mathcal{F}$ of mappings of $[0,1]$ into itself contains a map that is surjective, $\mathcal{F}$ satisfies the conditions of Theorem 4.2. We make use of this fact in the following theorem.

Theorem 4.3. *If $f : I \to 2^I$ is an upper semi-continuous function that is the union of a finite collection $\mathcal{F}$ of mappings of $[0,1]$ into itself and at least one member of $\mathcal{F}$ is surjective, then $\varprojlim \boldsymbol{f}$ is a one-dimensional continuum that contains a copy of every inverse limit $\varprojlim \boldsymbol{g}$ where $g_i \in \mathcal{F}$ for each i.*

Proof: Theorem 3.3 yields that $\varprojlim \boldsymbol{f}$ is a continuum. That the dimension of the inverse limit is 1 follows from Theorem 2.1. □

Richard M. Schori [9] constructed a chainable continuum that contains a copy of every chainable continuum. Although that is much stronger than our next theorem, we still observe this corollary to Theorem 3.3.

Corollary 4.4. *There exists an upper semi-continuous function f such that $\varprojlim \boldsymbol{f}$ is a one-dimensional continuum that contains a copy of every chainable continuum.*

Proof: There exist two mappings, $\varphi : [0,1] \twoheadrightarrow [0,1]$ and $\psi : [0,1] \to [0,1]$, where φ is surjective such that if M is a chainable continuum then there exists a sequence $\boldsymbol{k}$ such that $k_i \in \{\varphi, \psi\}$ for each i and M is homeomorphic to $\varprojlim \boldsymbol{k}$; see [1] (or [10] where it is shown that there is a map $g : [0,1] \twoheadrightarrow [0,1]$ such that φ may be chosen to be g and ψ may be chosen to be $g/2$). Let $f = \varphi \cup \psi$ and apply Theorem 4.3. □

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