Houston Journal of Mathematics

Volume 32, No. 1, 2006

INVERSE LIMITS OF UPPER SEMI-CONTINUOUS SET VALUED FUNCTIONS

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Communicated by Charles Hagopian

Abstract. In this article we define the inverse limit of an inverse sequence $(X_1, f_1), (X_2, f_2), (X_3, f_3), \ldots$ where each X_i is a compact Hausdorff space and each f_i is an upper semi-continuous function from X_{i+1} into 2^{X_i}. Conditions are given under which the inverse limit is a Hausdorff continuum and examples are given to illustrate the nature of these inverse limits.

1. Introduction

Inverse limits on closed subsets M of the unit square $[0, 1] \times [0, 1]$ were introduced in [4]. In this article we generalize this definition to inverse limit sequences $(X_1, f_1), (X_2, f_2), (X_3, f_3), \ldots$ where each X_i is a compact Hausdorff space and each f_i is an upper semi-continuous function from X_{i+1} into 2^{X_i}. Most of the theorems from [4] extend directly to this case with minor modifications and these extensions were announced in [4]. We also establish some mapping theorems between these inverse limits, provide some examples to indicate the variety of continua that can be produced, and provide some examples to answer some of the questions raised in [4].

2. Definitions and Notation.

If Y is a compact Hausdorff space, then 2^Y is the hyperspace of compact subsets of Y. Let each of X and Y be a compact Hausdorff space and let f be a function from X into 2^Y. The function f is *upper semi-continuous* at the point $x \in X$ if and only if for each open set V in Y containing $f(x)$, there is an open set U in X containing x such that if u is in U, then $f(u) \subseteq V$. The *graph*

2000 *Mathematics Subject Classification.* 54C60, 54B10, 54D80.

Key words and phrases. Inverse limits, set-valued functions.

$G(f)$ of f is the set of all points (x, y) such that y is in $f(x)$. A *mapping*, (or, for short, a *map*) is a continuous function. Inverse limits are usually defined for sequences $(X_1, f_1), (X_2, f_2), (X_3, f_3), \ldots$ such that for each i X_i is a topological space and f_i is a mapping from X_{i+1} into X_i. Such a sequence is called an *inverse limit sequence* and the mappings f_i are called *bonding maps*. In this article we consider sequences $(X_1, f_1), (X_2, f_2), (X_3, f_3), \ldots$ such that for each i X_i is a compact Hausdorff space and f_i is an upper semi-continuous function from X_{i+1} into 2^{X_i}. We again refer to such a sequence as an *inverse limit sequence* and the functions as *bonding functions*. In this case the inverse limit of the sequence $(X_1, f_1), (X_2, f_2), (X_3, f_3), \ldots$ is a subspace of $\Pi = \prod_{i>0} X_i$ with the product topology. The points of the inverse limit are the sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ in Π such that x_i is in $f_i(x_{i+1})$ for each i. If $f_i(x)$ is degenerate for each i and each point x of X_{i+1} then this definition reduces to the usual one. In this article we use bold characters to denote sequences and roman or italic characters to denote the terms of the sequence. Thus if $(X_1, f_1), (X_2, f_2), (X_3, f_3), \ldots$ is an inverse limit sequence with upper semi-continuous bonding functions, then $\mathbf{f} = f_1, f_2, f_3 \ldots$ and we denote the inverse limit by $\varprojlim \mathbf{f}$. As usual in inverse limits we use π_i to denote the projection of a point $\mathbf{x}$ in Π onto the i^{th} coordinate space X_i so that $\pi_i(\mathbf{x}) = x_i$. In [4] a closed subset of the unit square whose x-projection is $[0, 1]$ was shown to be the graph of upper semi-continuous functions from $[0, 1]$ into $2^{[0,1]}$. The next theorem shows that this result extends to compact Hausdorff spaces and that the graph G(f) of an upper semi-continuous function $f : X \to 2^Y$ is closed. As a consequence of the following theorem if X is a compact Hausdorff space and M is a closed subset of $X \times X$, then there is an upper semi-continuous function $f : X \to 2^X$ such that $G(f) = M$ so we may define $\varprojlim \mathbf{M}$ to be $\varprojlim \mathbf{f}$ where $\mathbf{f} = f, f, f, \ldots$.

Theorem 2.1. *Suppose each of X and Y is a compact Hausdorff space and M is a subset of $X \times Y$ such that if x is in X then there is a point y in Y such that (x, y) is in M. Then M is closed if and only if there is an upper semi-continuous function $f : X \to 2^Y$ such that $M = G(f)$.*

Proof. A proof that M is closed if M is the graph $G(f)$ of an upper semi-continuous function f from X into 2^Y can be found in [3, Theorem 1, p. 175].

Assume that M is closed and for each x in X, define $f(x)$ to be $\{y \in Y \mid (x, y) \in M\}$. Since M is closed, $f(x)$ is closed for each x in X. To see that f is upper semi-continuous, suppose x is in X and V is an open set in Y containing $f(x)$. If f is not upper semi-continuous at x, then for each open set U containing x there exist points z of U and (z, y) of M such that y is not in V. For each open set U containing x, denote by M_U the set of all points (p, q) of M such that p is in $\overline{U}$ and q is not in V. Since the collection of all the closed sets M_U has the finite intersection property and $X \times Y$ is compact, there is a point (a, b) common to all the sets M_U. Since each M_U is a subset of M, (a, b) belongs to M. Since x is the only point common to all the sets $\overline{U}$, $a = x$. Further, b is not in V. This contradicts the fact that b belongs to $f(x)$. □

3. Compact inverse limits

In this section we assume that $(X_1, f_1), (X_2, f_2), (X_3, f_3), \ldots$ is an inverse limit sequence with upper semi-continuous bonding functions, and that X_i is a compact Hausdorff space for each i. It will be convenient to introduce the following notation: if n is a positive integer, G_n denotes the set of all points $\mathbf{x}$ of $\Pi = \prod_{i>0} X_i$ such that $x_i \in f_i(x_{i+1})$ for $i \leq n$.

Theorem 3.1. *For each positive integer n, G_n is a non-empty compact set.*

Proof. Since Π is compact it suffices to show that G_n is closed. Let $\mathbf{x}$ be a point of Π that is not in G_n. There is a positive integer $k \leq n$ such that x_k is not in $f_k(x_{k+1})$. But $f_k(x_{k+1})$ is compact and closed since X_k is a compact Hausdorff space. Thus there are mutually exclusive open sets O and U in X_k that contain x_k and $f_k(x_{k+1})$ respectively. As f_k is upper semi-continuous, there is an open set V containing x_{k+1} such that if t is in V, then $f_k(t) \subseteq U$. From this it follows that $\pi_k^{-1}(O) \cap \pi_{k+1}^{-1}(V)$ is an open set in Π that contains $\mathbf{x}$ but no point of G_n so G_n is closed and therefore compact. To see that G_n is non-empty, let $\mathbf{y}$ be a point of G_n whose coordinates are defined inductively as follows. Select a point y_{n+1} of X_{n+1} and let y_n be a point of $f_n(y_{n+1})$. Continue to inductively define y_{n-i} to be a point of $f_{n-i}(y_{n-i+1})$ for $i < n$. For $i > n+1$ let y_i be any point of X_i. □

As an immediate consequence of Theorem 3.1 we have the result we sought in the following theorem.

Theorem 3.2. $K = \varprojlim \mathbf{f}$ *is non-empty and compact.*

Proof. $G_1, G_2, G_3, \ldots$ is a nested sequence of non-empty compact sets in the compact Hausdorff space Π, so $\bigcap_{i>0} G_i$ is non-empty and compact. Clearly $K = \bigcap_{i>0} G_i$. □

If for each i and each point x of X_i there is a point y of X_{i+1} such that $x \in f_i(y)$ then for each point x_1 of X_1 there is a point $\mathbf{x}$ in the inverse limit with $\pi_1(\mathbf{x}) = x_1$. Thus, in this case, one does not need the preceding theorem to see that the inverse limit is non-empty. This is analogous to the case for inverse limits with surjective mappings.

4. Connected inverse limits

We next turn our attention to conditions under which inverse limits are connected. By a *Hausdorff continuum* we mean a compact, connected subset of a Hausdorff space while by a *continuum* we mean a compact, connected subset of a metric space. We begin by giving conditions under which $G(f)$ is connected when f is an upper semi-continuous function from the compact Hausdorff space X into closed subsets of a compact Hausdorff space Y. The fact that $G(f)$ is connected does not imply that the the inverse limit with such a function as the only bonding function is connected, even when each of X and Y is the interval $[0, 1]$. See [4, Example 1] and Example 1 of Section 6 of this article.

Theorem 4.1. *Suppose that each of X and Y is a compact Hausdorff space, X is connected, f is an upper semi-continuous function from X into 2^Y, and for each x in X $f(x)$ is connected. Then the graph $G(f)$ of f is connected.*

Proof. Recall that $G(f) = \{(x, y) \in X \times Y \mid y \in f(x)\}$ and assume that $G(f)$ is not connected. There are then two non-empty mutually separated sets H and K with union $G(f)$. The sets H and K are *mutually separated* if $H \cap \overline{K} = K \cap \overline{H} = \emptyset$. If x is in X, then $\{x\} \times f(x)$ is a connected subset of $G(f)$ and thus a subset of one of H or K. Let H_1 be the set of all points x of X such that $\{x\} \times f(x)$ lies in H and let K_1 be the points x of X such that $\{x\} \times f(x)$ lies in K. H_1 and K_1 are non-empty compact sets whose union is the connected set X so they have a common point z. But this is impossible since $\{z\} \times f(z)$ would then be a connected subset of both H and K. □

If M is a subset of the product $X \times Y$ of compact Hausdorff spaces, then the inverse of M is the subset of $Y \times X$ consisting of all points $(y, x) \in Y \times X$ such that (x, y) is in M. We denote this inverse by M^{-1}. The preceding theorem for Hausdorff continuum-valued, upper semi-continuous functions has a counterpart for upper semi-continuous functions whose graphs have inverses that are the graphs of upper semi-continuous Hausdorff continuum-valued functions.

Theorem 4.2. *Suppose that X and Y are compact Hausdorff spaces, Y is connected, f is an upper semi-continuous function from X into 2^Y such that for each y in Y $\{x \in X \mid y \in f(x)\}$ is a non-empty, connected set. Then $G(f)$ is connected.*

PROOF. Let $M = G(f)^{-1}$. Observe that M is closed if and only if M^{-1} is closed. Since by Theorem 2.1 M^{-1} is the graph of an upper semi-continuous, Hausdorff continuum-valued function from Y into 2^X, M^{-1} is connected by Theorem 4.1. $\square$

For the next theorem it will be convenient to generalize the definition of the graph $G(f)$ of an upper semi-continuous function. If $Y_1, Y_2, \ldots, Y_{m+1}$ is a finite collection of compact Hausdorff spaces and $g_i : Y_{i+1} \to 2^{Y_i}$ is an upper semi-continuous function for $1 \leq i \leq m$, let $G(g_1, g_2, \ldots, g_m) = \{x \in \prod_{i=1}^{m+1} Y_i \mid x_i \in g_i(x_{i+1}), 1 \leq i \leq m\}$. If $m = 1$, then $G(g_1)$ is the graph of g_1.

Theorem 4.3. *Suppose $X_1, X_2, \ldots, X_{n+1}$ is a finite collection of Hausdorff continua and $f_1, f_2, \ldots, f_n$ is a finite collection of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ for $1 \leq i \leq n$. If $f_i(x)$ is connected for each x in X_{i+1} and each $i, 1 \leq i \leq n$, then $G(f_1, f_2, \ldots, f_n)$ is connected.*

PROOF. We proceed by induction on the number of arguments for G. The case that there is only one argument is Theorem 4.1. Suppose the theorem holds for n arguments and let $f_1, f_2, \ldots, f_{n+1}$ be $n+1$ arguments for G. By the inductive hypothesis, $G(f_2, \ldots, f_{n+1})$ is connected. If H and K are closed sets whose union is $G(f_1, \ldots, f_{n+1})$ and $h : G(f_1, \ldots, f_{n+1}) \to G(f_2, \ldots, f_{n+1})$ is the continuous transformation defined by $h(x_1, \ldots, x_{n+1}) = (x_2, \ldots, x_{n+1})$, then $h(H \cup K) = G(f_2, \ldots, f_{n+1})$. Thus, there is a point p belonging to $h(H)$ and $h(K)$. Then, $\{x \in G(f_1, \ldots, f_{n+1}) \mid x_1 \in f_1(p_2)$ and $x_i = p_i$ for $2 \leq i \leq n+1\}$ is a connected set intersecting both H and K. Consequently, H and K are not mutually separated. $\square$

The next theorem is an immediate consequence of Theorem 4.3.

Theorem 4.4. *Assume that for each i X_i is a Hausdorff continuum and for each $x \in X_{i+1}$ $f_i(x)$ is connected. Then for each positive integer n, G_n is connected.*

PROOF. Since $G_n = G(f_1, f_2, \ldots, f_n) \times X_{n+2} \times X_{n+3} \times \cdots$ this follows immediately from Theorem 4.3. $\square$

Recall that Theorem 4.1 and Theorem 4.2 were closely related, one being for Hausdorff continuum-valued upper semi-continuous functions and the other for functions whose graphs had inverses that were the graphs of Hausdorff continuum-valued upper semi-continuous functions. The next two theorems bear a similar relation to Theorem 4.3 and Theorem 4.4.

Theorem 4.5. *Suppose $X_1, X_2, \ldots, X_{n+1}$ is a finite collection of Hausdorff continua and $f_1, f_2, \ldots, f_n$ is a finite collection of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ for $1 \leq i \leq n$. If for each $i, 1 \leq i \leq n$ and each y of $X_i, \{x \in X_{i+1} \mid y \in f_i(x)\}$ is a non-empty, connected set, then $G(f_1, f_2, \ldots, f_n)$ is connected.*

Proof. We proceed by induction. The case $n = 1$ is Theorem 4.2. Assume the theorem holds for n, and let $f_1, \ldots, f_{n+1}$ be a collection of $n+1$ arguments for G. If H and K are closed sets whose union is $G(f_1, \ldots, f_{n+1})$ and $h : G(f_1, \ldots, f_{n+1}) \to G(f_1, \ldots, f_n)$ is the continuous transformation defined by $h(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$, then $h(H) \cup h(K) = G(f_1, \ldots, f_n)$, so there is a point p belonging to $h(H)$ and $h(K)$. Then, $\{x \in G(f_1, \ldots, f_{n+1}) \mid x_i = p_i, 1 \leq i \leq n$, and $x_n \in f_n(x_{n+1})\}$ is a connected set intersecting both H and K. Thus H and K are not mutually separated. □

As an immediate corollary we have the following theorem.

Theorem 4.6. *Assume that for each i X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X^i}$ is an upper semi-continuous function, and for each $x_i \in X_i$ $\{y \in X_{i+1} \mid x_i \in f_i(y)\}$ is a non-empty, connected set. Then for each positive integer n, G_n is connected.*

As a consequence of the preceding theorems, in the next two theorems we have the results we sought in this section.

Theorem 4.7. *Suppose that for each i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X^i}$ is an upper semi-continuous function, and for each x in X_{i+1}, $f_i(x)$ is connected. Then $\varprojlim \mathbf{f}$ is a Hausdorff continuum.*

Proof. For each i, we have from Theorem 3.1 that G_i is compact and from Theorem 4.4 that G_i is connected. So $\varprojlim \mathbf{f} = \bigcap_{i>0} G_i$ is a Hausdorff continuum. □

Theorem 4.8. *Suppose that for each i X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X^i}$ is an upper semi-continuous function, and for each $x \in X_i$ $\{y \in X_{i+1} \mid x \in f_i(y)\}$ is a non-empty, connected set. Then $\varprojlim \mathbf{f}$ is a Hausdorff continuum.*

5. Mapping theorems

Suppose $X_1, X_2, X_3, \ldots$ is a sequence of compact Hausdorff spaces and $n_1, n_2, n_3, \ldots$ is an increasing sequence of positive integers. The function $F : \prod_{i>0} X_i \to \prod_{i>0} X_{n_i}$ given by $F(x_1, x_2, x_3, \ldots) = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ is continuous. If, in addition, for each i, $f_i : X_{i+1} \to X_i$ is a mapping and

$g_i = f_{n_i} \circ f_{n_i+1} \circ \cdots \circ f_{n_{i+1}-1}$, then it is well known that the restriction of this map to $\varprojlim \mathbf{f}$ is a homeomorphism onto $\varprojlim \mathbf{g}$.

For inverse limits of the type considered in this paper, the situation is similar but $F \mid \varprojlim \mathbf{f}$ need not be a homeomorphism. If X, Y, and Z are compact Hausdorff spaces and $f : X \to 2^Y$ and $g : Y \to 2^Z$ are upper semi-continuous functions, we define $g \circ f : X \to 2^Z$ by $(g \circ f)(x) = \{z \in Z \mid$ there is a point y of Y such that $y \in f(x)$ and $z \in f(y)\}$. We include the following theorem for the sake of completeness.

Theorem 5.1. *Suppose $X_1, X_2, X_3, \ldots$ is a sequence of compact Hausdorff spaces and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function for each positive integer i. If $n_1, n_2, n_3, \ldots$ is an increasing sequence of positive integers let $g_1, g_2, g_3, \ldots$ be the sequence such that $g_i = f_{n_i} \circ f_{n_i+1} \circ \cdots \circ f_{n_{i+1}-1}$. If $F : \prod_{i>0} X_i \to \prod_{i>0} X_{n_i}$ given by $F(x_1, x_2, x_3, \ldots) = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ then $F \mid \varprojlim \mathbf{f}$ is a continuous transformation from $\varprojlim \mathbf{f}$ onto $\varprojlim \mathbf{g}$.*

Example 3 of the next section shows that the mapping of inverse limits from Theorem 5.1 need not be a homeomorphism, even in the special case where each $f_i = f$ and each $g_i = f \circ f$.

Suppose $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$ are sequences of compact Hausdorff spaces and, for each positive integer i, $f_i : X_{i+1} \to 2^{X_i}$ and $g_i : Y_{i+1} \to 2^{Y_i}$ are upper semi-continuous functions. Suppose further that, for each positive integer i, $\varphi_i : X_i \to Y_i$ is a mapping. The function $\Phi : \prod_{i>0} X_i \to \prod_{i>0} Y_i$ given by $\Phi(\mathbf{x}) = (\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3), \ldots)$ is continuous and Φ is one-to-one if each φ_i is one-to-one. These observations lead to the following theorem.

Theorem 5.2. *Suppose $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$ are sequences of compact Hausdorff spaces and, for each positive integer i, $f_i : X_{i+1} \to 2^{X_i}$ and $g_i : Y_{i+1} \to 2^{Y_i}$ are upper semi-continuous functions. Suppose further that, for each positive integer i, $\varphi_i : X_i \to Y_i$ is a mapping such that $\varphi_i \circ f_i = g_i \circ \varphi_{i+1}$. The function $\varphi : \varprojlim \mathbf{f} \to \varprojlim \mathbf{g}$ given by $\varphi(\mathbf{x}) = (\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3), \ldots)$ is continuous. Furthermore, φ is one-to-one (and surjective) if each φ_i is one-to-one (and surjective).*

Proof. In light of the observations preceding the statement of this theorem, since $\varphi = \Phi \mid \varprojlim \mathbf{f}$ there are only a couple of things to check. First, we need to show that if $\mathbf{x}$ is in $\varprojlim \mathbf{f}$ then $\varphi(\mathbf{x})$ is in $\varprojlim \mathbf{g}$. This involves checking that $\varphi_i(x_i) \in g_i(\varphi_{i+1}(x_{i+1}))$ for each positive integer i. However, this follows from the fact that $\varphi_i(f_i(x_{i+1})) = g_i(\varphi_{i+1}(x_{i+1}))$. So, $\varphi(\varprojlim \mathbf{f})$ is a subset of $\varprojlim \mathbf{g}$. The only other thing to check is that φ is surjective whenever each φ_i is one-to-one and surjective. Suppose that $\mathbf{y}$ is in $\varprojlim \mathbf{g}$. The point $\mathbf{x} = (\varphi_1^{-1}(y_1), \varphi_2^{-1}(y_2), \varphi_3^{-1}(y_3), \ldots)$ belongs to $\prod_{i>0} X_i$ and $\varphi(\mathbf{x}) = \mathbf{y}$. Thus, we only need to see that $\mathbf{x}$ is in $\varprojlim \mathbf{f}$. To this end, let i be a positive integer and consider $f_i(x_{i+1})$. Since $y_{i+1} = \varphi_{i+1}(x_{i+1})$ and $y_i \in g_i(y_{i+1})$, we have $y_i \in g_i(\varphi_{i+1}(x_{i+1})) = \varphi_i(f_i(x_{i+1}))$. Therefore, there is a point t of $f_i(x_{i+1})$ so that $\varphi_i(t) = y_i$. Since φ_i is one-to-one, $t = x_i$. □

Suppose X is a compact Hausdorff space. If $f : X \to 2^X$ and $g : X \to 2^X$ are upper semi-continuous functions, f and g are *topologically conjugate* provided there is a homeomorphism h such that $h(X) = X$ and $h \circ f = g \circ h$. We have the following corollary to Theorem 5.2.

Theorem 5.3. *Suppose X is a compact Hausdorff space. If $f : X \to 2^X$ and $g : X \to 2^X$ are topologically conjugate upper semi-continuous functions, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$.*

6. The special case where each $X_i = [0, 1]$.

In this section we provide some examples in the special case where each X_i is $I = [0, 1]$ and we have a single bonding function. If f is an upper semi-continuous function from I into 2^I, then $\varprojlim \mathbf{f}$ will denote the inverse limit of the inverse sequence $(I, f), (I, f), (I, f), \ldots$. In view of Theorem 2.1 we can consider closed subsets of $I \times I$ whose projection on the x-axis is I since each such set is the graph of an upper semi-continuous function. If M is a closed subset of $I \times I$ whose projection on the x-axis is I and f is the corresponding upper semi-continuous function, then $M = G(f)$. In the following examples we follow the convention used in [4], and define $\varprojlim \mathbf{M}$ to be $\varprojlim \mathbf{f}$. In this case $\varprojlim \mathbf{M}$ is a subset of the Hilbert cube $\mathcal{Q}$.

In [4] an example was given of a closed subset M of $I \times I$ such that M is connected but $\varprojlim \mathbf{M}$ is not connected. In that example the projection of M on the y-axis was not I. The following provides an example where the projection on the y-axis is I.

Example 1. *Let M be the union of the four straight line intervals, $I \times \{0\}$, $\{1\} \times I$, the interval from $(0, 0)$ to $(1/4, 1/4)$, and the interval from $(3/4, 1/4)$ to $(1, 1)$.*

Let N be the set of all points $\mathbf{p}$ of $K = \varprojlim \mathbf{M}$ such that $p_1 = p_2 = 1/4$ and $p_3 = 3/4$. Let $\mathbf{x}$ be a point of N. Let $R = R_1 \times R_2 \times R_3 \times \mathcal{Q}$ be the region in $\mathcal{Q}$ where $R_1 = R_2 = (1/8, 3/8)$ and $R_3 = (5/8, 7/8)$, and note that R contains $\mathbf{x}$.

Assume that the point $\mathbf{y}$ is in $R \cap K$. Then y_1 and y_2 are in $(1/8, 3/8)$. It follows that $y_2 \leq 1/4$. But if $y_2 < 1/4$, then $y_3 = 1$ and $\mathbf{y}$ is not in R. We conclude that R contains no point not in N and that N and $K - N$ are mutually separated and K is not connected.

To illustrate the variety of continua one can obtain we include a simple example whose inverse limit is a fan.

Example 2. *Let M be the union of the graph of the identity function and the interval $I \times \{0\}$.*

Note that if $\mathbf{p} \in K = \varprojlim \mathbf{M}$ and for some n, $p_n > 0$, then $p_j = p_n$ for all $j > n$. For each positive integer n, let K_n be the set of all points $\mathbf{p} \in \varprojlim \mathbf{M}$ such that $p_j = 0$ for $j < n$ and $p_j = p_n > 0$ for $j \geq n$. The closure of K_n is an arc of length $1/2^{n-1}$ having one endpoint at $(0, 0, 0, \ldots)$. Moreover no two of these arcs intersect except at their common endpoint and $\varprojlim \mathbf{M} = \bigcup_{i>0} K_i$.

It follows from Theorem 5.1 of Section 5 that for an upper semi-continuous function $f : I \to 2^I$, there is a map from $\varprojlim \mathbf{f}$ onto $\varprojlim \mathbf{f^2}$. It is well known that if f is a map from I into I, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{f^2}$. The following example shows that this fact does not generalize.

Example 3. *Let M be the union of the three straight line intervals $I \times \{1/2\}$, $\{1\} \times [0, 1/2]$ and the interval with endpoints $(0, 1)$ and $(1/2, 1/2)$.*

If f is the upper semi-continuous function determined by M, then $K = \varprojlim \mathbf{f}$ contains a triod which is the union of the three arcs which are described below. Let A_1 be the set of all points of K whose first coordinate is in the half open interval $(1/2, 1]$. The closure of A_1 is an arc from $(1, 0, 1, 0, \ldots)$ to $(1/2, 1/2, 1, 0, 1, 0, \ldots)$. Let A_2 be the set of all points of K whose first two coordinates are $1/2$ and whose third coordinate is in the half open interval $(1/2, 1]$. The closure of A_2 is an arc from $(1/2, 1/2, 1/2, 1/2, 1, 0, 1, 0, \ldots)$ to $(1/2, 1/2, 1, 0, 1, 0, \ldots)$. Finally let A_3 be the set of all points of K whose first coordinate is $1/2$ and whose second coordinate is in the half open interval $[0, 1/2)$. The closure of A_3 is an arc from $(1/2, 0, 1, 0, 1, 0, \ldots)$ to $(1/2, 1/2, 1, 0, 1, 0, \ldots)$. The union of the closures of A_1, A_2, and A_3 is a simple triod contained in K.

On the other hand, $f \circ f$ is the union of the three straight line intervals, $\{0\} \times [0, 1/2]$, $I \times \{1/2\}$ and $\{1\} \times [1/2, 1]$. $K = \varprojlim \mathbf{f} \circ \mathbf{f}$ is a continuum by Theorem 4.7. We will show that K is an arc with endpoints $a = (0, 0, 0, \ldots)$ and $b = (1, 1, 1, \ldots)$. Let $\mathbf{p}$ be a point of K different from a and b. There is an n such that p_n is neither 0 nor 1. If $p_n \neq 1/2$ then $K \cap \pi_n^{-1}(p_n)$ is degenerate and separates

K into the two mutually separated sets $K \cap \pi_n^{-1}([0, p_n))$ and $K \cap \pi_n^{-1}((p_n, 1])$. If $p_n = 1/2$ and $p_{n+1} \in \{0, 1\}$ then $K \cap \pi_n^{-1}(p_n)$ is degenerate and a separating point (i.e., cut point) of K. Thus we may assume that p_{n+1} is neither 0 nor 1 and again conclude that $K \cap \pi_{n+1}^{-1}(p_{n+1})$ is a separating point unless $p_{n+1} = 1/2$. Continuing this process, the only point remaining to consider is the constant sequence $(1/2, 1/2, 1/2, \ldots)$. But this point also is clearly a separating point of K. Thus K is a continuum having at most two non-separating points and is an arc. See [1, Theorem 1-18, p. 49] and [1, Theorem 2-27, p. 54].

We next provide an example of a closed set M in I^2 such that if f is the upper semi-continuous function determined by M, then not only is $\varprojlim \mathbf{f}$ not homeomorphic to $\varprojlim \mathbf{f} \circ \mathbf{f}$ but it also is a well known example of a universal continuum. That is, it is a continuum with the property that if K is a continuum, then K is the image of a subcontinuum of M under a continuous mapping.

Example 4. *Let M be the union of the four straight line intervals joining the point $(0, 1/2)$ to $(1/2, 1)$, $(1/2, 1)$ to $(1, 1/2)$, $(1, 1/2)$ to $(1/2, 0)$, and $(1/2, 0)$ to $(0, 1/2)$.*

Note that the four arcs whose union is M form a diamond in I^2. Label these arcs A_i for $i \in \{1, 2, 3, 4\}$ in a clockwise direction so that $A_1 \subset [0, 1/2] \times [1/2, 1]$ and $A_4 \subset [0, 1/2] \times [0/1/2]$. Let f be the upper semi-continuous function determined by M and let $K = \varprojlim \mathbf{f}$. The set K contains a simple closed curve which is the union of the four arcs B_i for $i \in \{1, 2, 3, 4\}$ determined as follows. If $i = 2$ or $i = 4$, B_i is the set of all points $\mathbf{p} \in K$ such that for each n, $(p_{n+1}, p_n) \in A_i$. If $i = 1$ or 3, then B_i is the set of all points $\mathbf{p} \in K$ such that for each odd n, $(p_{n+1}, p_n) \in A_i$ and for each even n $(p_{n+1}, p_n) \in A_{i+2(mod 4)}$.

On the other hand, the graph of $f \circ f$ is the union of the two arcs, one from $(0, 0)$ to $(1, 1)$ and the other from $(0, 1)$ to $(1, 0)$ and $\varprojlim \mathbf{f} \circ \mathbf{f}$ is homeomorphic to the cone over a Cantor set as was shown in [4].

Perhaps of more interest as indicated above is the continuum $K = \varprojlim \mathbf{f}$. K contains two mutually exclusive Cantor sets, C_0 consisting of all points $\mathbf{p}$ of K such that $p_n = 1/2$ if n is even and C_1 consisting of all points $\mathbf{p}$ of K such that $p_n = 1/2$ if n is odd. If $\mathbf{a}$ is a point of C_0 and $\mathbf{b}$ is a point of C_1, then for each n (a_{n+1}, a_n) and (b_{n+1}, b_n) are endpoints of the arc A_{i_n} where $i_n \in \{1, 2, 3, 4\}$. The set of all points $\mathbf{x}$ of K such that (x_{n+1}, x_n) is in A_{i_n} is an arc joining $\mathbf{a}$ and $\mathbf{b}$. K is the union of all these arcs, no two of which have a point in common that is not an endpoint. This is the example of a universal continuum given by Hurewicz in [2]. In fact, Hurewicz showed that if C is a continuum then there exist a subcontinuum H of K and a *monotone* map of H onto C.

In [4] it was conjectured that for closed subsets of $I \times I$ the inverse limit would be either 1-dimensional or infinite dimensional. This is not only wrong, but for every positive integer n there is a closed subset of $I \times I$ such that the corresponding inverse limit is n-dimensional. The next example provides a closed subset of I^2 whose inverse limit is 2-dimensional. We have been unable to find an example of a closed subset M of I^2 such that $\varprojlim \mathbf{M}$ is a 2-cell.

Example 5. *Let M consist of the union of the four straight line intervals, $[0, 1/2] \times \{0\}$, $\{1/2\} \times [0, 1/2]$, $[1/2, 1] \times \{1/2\}$ and $\{1\} \times [1/2, 1]$.*

As usual, let f be the upper semi-continuous function determined by M and $K = \varprojlim \mathbf{f}$. Here K is the union of a 2-cell D and an arc A. To identify D, let i, j be positive integers with $j > i + 1$ and let $D_{i,j}$ be the 2-cell $\{\mathbf{p} \in K \mid p_i \in [0, 1/2], p_j \in [1/2, 1], p_k = 0 \text{ if } k < i, p_k = 1/2 \text{ if } i < k < j, p_k = 1 \text{ if } k > j\}$. Let D be the 2-cell that is the closure of the union of all the 2-cells $D_{i,j}$ where $i \geq 1$ and $j > i + 1$. Let $A = \{\mathbf{p} \in K \mid p_1 \in [1/2, 1], p_k = 1 \text{ if } k > 1\} \cup \{\mathbf{p} \in K \mid p_1 = 1/2, p_2 \in [1/2, 1], p_k = 1 \text{ if } k > 2\}$. Then, $K = D \cup A$ and $D \cap A = \{(1/2, 1/2, 1, 1, \ldots)\}$.

One can alter the previous example to produce an inverse limit of dimension n for any choice of n. For example, to produce an inverse limit of dimension 3 add a second stairstep between $1/4$ and $1/2$. That is, let M be the union of the intervals $[0, 1/4] \times \{0\}$, $\{1/4\} \times [1/4, 1/2]$, $[1/4, 1/2] \times \{1/4\}$, $\{1/2\} \times [1/4, 1/2]$, $[1/2, 1] \times \{1/2\}$ and $\{1\} \times [1/2, 1]$. Additional stairsteps can be added to produce higher dimensional inverse limits. We understand from private correspondence that Mr. Antonio Peláez has independently produced different examples of closed subsets of $I \times I$ with n-dimensional inverse limits.

References

[1] J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, MA, 1961 (now available from Dover Publications, New York).

[2] Witold Hurewicz, Über oberhalb-stetige Zerlegungen von Punktmengen in Kontinua, *Fund. Math.* **15** (1930), 57–60.

[3] C. Kuratowski, *Topology*, vol. 1, Academic Press, New York, 1966.

[4] William S. Mahavier, Inverse limits with subsets of $[0, 1] \times [0, 1]$, *Topology and Its Applications*, **141/1-3**, 2004, pp. 225-231.

Received February 4, 2004
Revised version received June 6, 2004

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