INVERSE LIMITS WITH SET-VALUED FUNCTIONS HAVING GRAPHS THAT ARE ARCS

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Abstract. Itzok Banič and Judy Kennedy recently drew attention to a natural but largely unexplored field of study in the theory of inverse limits with set-valued functions, namely using bonding functions having graphs that are arcs. At the end of that paper they pose a question: If $f : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous function such that $G(f^n)$ is connected for each n and $G(f)$ is an arc, is $\varprojlim \boldsymbol{f}$ connected? In this paper we provide a negative answer to that question. We also include some additional examples, a shape theorem due to Van Nall, and pose several questions concerning inverse limits with functions whose graph is an arc.

1. Introduction

One of the fundamental questions in the theory of inverse limits with upper semi-continuous set-valued functions is the question of the connectedness of such an inverse limit. One known characterization of connectedness of inverse limits with set-valued functions is Theorem 1.1, see [10, Theorem 116, p. 85] where it is shown (without using surjectivity) that connectivity of the inverse limit follows from the connectedness of certain "approximations" (specifically, the approximations are the product of sets G'_n from Theorem 1.1 and the Hilbert cube). As can be seen from the function $f : [0,1] \to 2^{[0,1]}$ given by $f(t) = 0$ for $0 \leq t < 1$ and $f(t) = \{0, 1/2\}$, without surjectivity an inverse limit can be connected even if the sets G'_n from Theorem 1.1 are not all connected, [5, Example 1.8, pp. 8–9].

Theorem 1.1. *Suppose $X_1, X_2, X_3, \ldots$ a sequence of Hausdorff continua and $f_i : X_{i+1} \to 2^{X_i}$ is a surjective upper semi-continuous function for each positive integer i. Then, $\varprojlim \boldsymbol{f}$ is connected if and only if $G'_n = \{\boldsymbol{x} \in X_1 \times X_2 \times \cdots \times X_{n+1} \mid x_i \in f_i(x_{i+1})$ for each positive integer $i, 1 \leq i \leq n\}$ is connected for each positive integer n.*

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However, this theorem is often difficult to implement in practice, so the search for sufficient conditions on the nature of the bonding functions to ensure connectivity of the inverse limit continues. In particular, we are interested in conditions that are simple to verify. For instance, it is known that such inverse limits are Hausdorff continua in case the factor spaces are Hausdorff continua and the bonding functions are upper semi-continuous with connected values [10, Theorem 126, p. 90]. Numerous examples show that, in general, the hypothesis that the bonding functions have connected values cannot be weakened even if the factor spaces are all the interval $[0,1]$. Specifically, there exists a single surjective set-valued bonding function on $[0,1]$ with a connected graph (its graph happens to be an arc) having an inverse limit that is not connected [10, Example 114, p. 83]. We include another such example in Example 4.4 below. Without the surjectivity of the bonding function, an inverse limit with a set-valued function having a graph that is an arc can even be totally disconnected (see Example 4.1). Much research has been directed toward determining sufficient conditions on the bonding functions on a continuum (particularly on $[0,1]$, and even with a single bonding function) to ensure that the inverse limit is a continuum.

Some positive connectivity results for a single surjective upper semi-continuous function on $[0,1]$ include two theorems due to Van Nall.

Theorem 1.2. [5, Theorem 2.3, p. 16] *Suppose $f:[0,1]\to 2^{[0,1]}$ is a surjective upper semi-continuous function. Then, $\varprojlim \boldsymbol{f}$ is connected if and only if $\varprojlim \boldsymbol{f^{-1}}$ is connected.*

Theorem 1.3. [5, Theorem 2.11, p. 25] *If f is a surjective upper semi-continuous function on $[0,1]$ with a connected graph that is a union of the graphs of a collection of upper semi-continuous functions on $[0,1]$ each having connected values then $\varprojlim \boldsymbol{f}$ is connected.*

On the other hand, Nall observed that if $G(f^n)$ fails to be connected for some positive integer n, then the inverse limit is not connected [5, Theorem 2.3, p. 16]. Examples show that there are functions for which Nall's condition can be difficult to check; specifically, if n is a positive integer there is a function $f:[0,1]\to 2^{[0,1]}$ such that $G(f^n)$ is not connected but $G(f^i)$ is connected for each positive integer $i<n$ [5, Example 2.25, p. 42]. Other examples show that there are functions having a nonconnected inverse limit but for which $G(f^n)$ is connected for each positive integer n [5, Examples 2.26 and 2.27] (also, Example 4.2 below).

In a recent article, Iztok Banič and Judy Kennedy [1], considered connectivity questions for inverse limits on $[0,1]$ with a single set-valued bonding function whose graph is an arc. Their paper is perhaps the first in the literature to concentrate on this particular class of set-valued bonding functions. In their paper, Banič and Kennedy asked whether an inverse limit on $[0,1]$ is connected in case there is a single bonding function having a graph that is an arc and $G(f^n)$ is connected for each n. In Example 4.2 of this paper we provide a negative answer this question. Beyond that example, however, we believe it is important to draw attention to the compacta that are obtained as inverse limits on $[0,1]$ with set-valued functions having graphs that are arcs.

2. Definitions and Notation

A *compactum* is a compact metric space; a *continuum* is a connected compactum. If X is a compactum, 2^X denotes the collection of all compact subsets of X. If each of X and Y is a compactum, a function $f : X \to 2^Y$, herein denoted $f : X \nearrow Y$, is said to be *upper semi-continuous at the point x of X* provided that if V is an open subset of Y that contains $f(x)$ then there is an open subset U of X containing x such that if t is a point of U then $f(t) \subseteq V$. A function $f : X \to 2^Y$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of X. If $f : X \to 2^Y$ is a set-valued function, by the *graph* of f, denoted $G(f)$, we mean $\{(x,y) \in X \times Y \mid y \in f(x)\}$; if $f : X \nearrow Y$ and $g : Y \nearrow Z$, then $g \circ f : X \nearrow Z$ denotes the function given by $z \in g \circ f(x)$ if and only if there is a point y of Y such that $y \in f(x)$ and $z \in g(y)$. It is known that if X and Y are compacta and M is a subset of $X \times Y$ such that X is the projection of M to its set of first coordinates then M is closed if and only if M is the graph of an upper semi-continuous function [10, Theorem 2.1]. If $\boldsymbol{s} = s_1, s_2, s_3, \ldots$ is a sequence, we normally denote the sequence in boldface type and its terms in italics. Suppose $\boldsymbol{X}$ is a sequence of compacta and $f_n : X_{n+1} \nearrow X_n$ is an upper semi-continuous function for each $n \in \mathbb{N}$. By the *inverse limit* of $\boldsymbol{f}$, denoted $\varprojlim \boldsymbol{f}$, we mean $\{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for each positive integer $i\}$. If the sequences $\boldsymbol{X}$ and $\boldsymbol{f}$ are constant sequences (i.e., there are a compactum X and an upper semi-continuous function $f : X \nearrow X$ such that $X_i = X$ and $f_i = f$ for each $i \in \mathbb{N}$), we say that $\varprojlim \boldsymbol{f}$ is an inverse limit with only one bonding function. For the most part in this paper, we are concerned with inverse limits with only one bonding function where the compactum X is the interval $[0,1]$. If X is a compactum and M is a subset of X, for convenience, we denote the product of countably many copies of M by M^∞. If $\{X_a \mid a \in D\}$ is a collection of sets and A is a subset of D, we denote by π_A the natural projection of $\prod_{a \in D} X_a$ onto $\prod_{a \in A} X_a$.

A set traditionally used in the proof that $\varprojlim \boldsymbol{f}$ is nonempty and compact is $\{\boldsymbol{x} \in \prod_{k>0} X_k \mid x_i \in f_i(x_{i+1})$ for $1 \leq i \leq n\}$. Because this set was originally denoted G_n, we adopt and use throughout this article the notation $G'_n = \{\boldsymbol{x} \in \prod_{k=1}^{n+1} X_k \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$ for the projection of G_n into the product of the first $n+1$ factor spaces, i.e., $G'_n = \pi_{\{1,2,\ldots,n+1\}}(G_n)$. These sets G'_n are precisely the "approximations" used in Theorem 1.1 above whose connectedness characterizes the connectedness of the inverse limit. Alternatively, at times, we may denote G'_n by $G'(f_1, f_2, \ldots, f_n)$.

It is becoming increasingly more evident that understanding the topology of inverse limits is rather closely tied to understanding the sets G'_n. Beyond their use in the proof of the existence and compactness of the inverse limit, one indication of their utility lies in the following theorem that allows us to employ the powerful tools of inverse limits with mappings in their study. The theorem is little more than an observation and is quite easy to prove, but we refer the reader to [7, Section 4, pp. 59–60] for a more complete treatment. See also [9].

Theorem 2.1. *Suppose* $\boldsymbol{X}$ *is a sequence of compacta and* $f : X_{i+1} \nearrow X_i$ *is a surjective upper semi-continuous function for each positive integer* i. *Then,* $\varprojlim \boldsymbol{f}$ *is homeomorphic to the inverse limit on the sequence of spaces* $X_1, G'(f_1), G'(f_1, f_2), G'(f_1, f_2, f_3), \ldots$ *with bonding functions that are projection mappings.*

From this theorem it follows that properties of the factor spaces that are preserved in inverse limits with mappings are properties that an inverse limit with upper semi-continuous bonding functions inherits from the spaces G'_n. Such properties include, but are not limited to, chainability, treelikeness, atriodicity, acyclicity, dimension less than or equal to k, irreducibility, trivial shape, and indecomposability.

We close this section with a theorem of M. M. Marsh [12, Theorem 2.1, p. 244]. For the convenience of the reader we state and prove Marsh's theorem as we make use of it on $[0, 1]$.

Theorem 2.2. (**Marsh**) Suppose $f : [0, 1] \nearrow [0, 1]$ and $g : [0, 1] \to [0, 1]$ is a mapping such that $g^{-1} \subseteq G(f)$. If n is a positive integer, $\varprojlim \boldsymbol{f}$ contains a homeomorphic copy of G'_n.

Proof. The function $h : G'_n \to \varprojlim \boldsymbol{f}$ given by $h(\boldsymbol{x}) = (x_1, x_2, \ldots, x_{n+1}, g(x_{n+1}), g^2(x_{n+1}), \ldots)$ is a homeomorphism. □

3. Trivial shape

In this section we depart from our theme of inverse limits with set-valued functions having graphs that are arcs and prove a general result due to Van Nall. In [3, Theorem 2] W. Charatonik and R. Roe showed that trivial shape is preserved in inverse limits on finite-dimensional continua having trivial shape using upper-semicontinuous bonding functions that have values that are continua with trivial shape. In private correspondence with the author Nall observed that by modifying their techniques the condition that the bonding functions have values that are continua with trivial shape can be replaced by the condition that the bonding functions have inverses with values that are continua with trivial shape. Because of its potential importance, we provide a proof of Nall's theorem. This proof is comparable to the proof provided by Charatonik and Roe. We begin with a lemma.

Lemma 3.1. *Suppose X and Y are finite dimensional continua, $f : X \nearrow Y$ is a surjective upper semi-continuous function, A is a subcontinuum of Y, and $f^{-1}(y)$ is a continuum with trivial shape for each y in A. If $B = \{(y,x) \in A \times X \mid x \in f^{-1}(y)\}$, then A and B have the same shape.*

Proof. Note that $\pi_1 : B \twoheadrightarrow A$ is a mapping. Choose y in A and define $h : f^{-1}(y) \twoheadrightarrow \pi_1^{-1}(y)$ by $h(t) = (y,t)$ for each t in $f^{-1}(y)$. Then, h is $1-1$ and continuous so h is a homeomorphism. Thus, $\pi_1^{-1}(y)$ has trivial shape for each y in Y. By a theorem of R. B. Sher [14, Theorem 11, p. 86], A and B have the same shape. □

In Lemma 3.1 if the set A is Y then the set B is $G(f^{-1})$. Recalling that $G(f)$ and $G(f^{-1})$ are homeomorphic we have the following.

Theorem 3.2. *Suppose X and Y are finite dimensional continua, $f : X \nearrow Y$ is a surjective upper semi-continuous function and $f^{-1}(y)$ is a continuum with trivial shape for each y in Y. If Y has trivial shape, then $G(f)$ and $G(f^{-1})$ have trivial shape.*

Theorem 3.3. *Suppose $\boldsymbol{X}$ is a sequence of finite-dimensional continua with trivial shape, $f_i : X_{i+1} \nearrow X_i$ is a surjective upper semi-continuous function for each positive integer i, and $f_i^{-1}(x)$ is a continuum with trivial shape for each positive integer i and each $x \in X_i$. Then, $G'(f_1, f_2, \dots, f_n)$ has trivial shape for each positive integer n.*

Proof. Because $G'(f_1) = G(f_1^{-1})$, by Theorem 3.2, $G'(f_1)$ has trivial shape.

Inductively, suppose n is a positive integer such that $G'(f_1, f_2, \dots, f_n)$ has trivial shape. Let $f : G'(f_1, f_2, \dots, f_{n+1}) \to G'(f_1, f_2, \dots, f_n)$ be the projection mapping given by $f(\boldsymbol{x}) = (x_1, x_2, \dots, x_{n+1})$. Choose $\boldsymbol{y} = (y_1, y_2, \dots, y_{n+1})$ in $G'(f_1, f_2, \dots, f_n)$ and note that $h : f_{n+1}^{-1}(y_{n+1}) \twoheadrightarrow f^{-1}(\boldsymbol{y})$ given by $h(t) = (y_1, y_2, \dots, y_{n+1}, t)$ is a homeomorphism. Thus, $f^{-1}(\boldsymbol{y})$ has trivial shape for each $\boldsymbol{y} \in G'(f_1, f_2, \dots, f_n)$ so Sher's theorem [14] yields that $G'(f_1, f_2, \dots, f_{n+1})$ has trivial shape. □

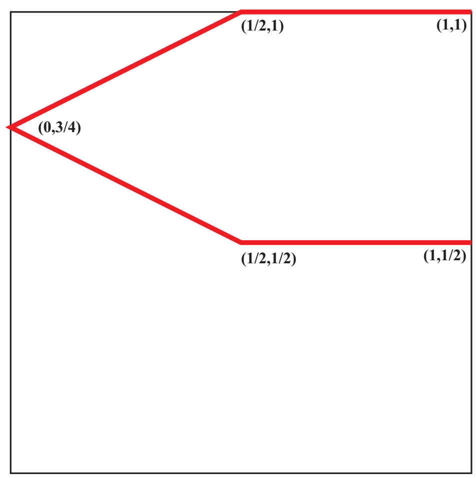

FIGURE 1. The graph of the Mahavier bonding function f in Example 4.1.

The following theorem now follows from Theorem 3.3 and Theorem 2.1.

Theorem 3.4. (**Nall**) Suppose $\boldsymbol{X}$ is a sequence of finite-dimensional continua with trivial shape and $f_i : X_{i+1} \nearrow X_i$ is upper semi-continuous for each positive integer i. If $f_i^{-1}(y)$ is a continuum with trivial shape for each $i \in \mathbb{N}$ and each $y \in Y$, then $\varprojlim \boldsymbol{f}$ has trivial shape.

4. Examples

In his article introducing inverse limits with set-valued functions, Bill Mahavier included an example of a function having an arc as its graph but whose inverse limit is a Cantor set [11, Example 2]. For completeness we sketch a proof.

Example 4.1. **(Mahavier)** *Let* $f : [0,1] \nearrow [0,1]$ *be the upper semi-continuous function whose graph consists of four straight line intervals, one from* $(0,3/4)$ *to* $(1/2,1)$*, one from* $(1/2,1)$ *to* $(1,1)$*, one from* $(0,3/4)$ *to* $(1/2,1/2)$*, and one from* $(1/2,1/2)$ *to* $(1,1/2)$*. Then* $G(f)$ *is an arc and* $\varprojlim \boldsymbol{f}$ *is a Cantor set.* (See Figure 1 for the graph of f).

Proof. It is not difficult to see that $\varprojlim \boldsymbol{f}$ contains the Cantor set $\{1/2,1\}^\infty$. Moreover, if $\boldsymbol{x} \in \varprojlim \boldsymbol{f}$, no coordinate of $\boldsymbol{x}$ is less than $1/2$; for this reason it follows that no coordinate of $\boldsymbol{x}$ can be strictly between $1/2$ and 1. Thus $\varprojlim \boldsymbol{f} = \{1/2,1\}^\infty$. $\square$

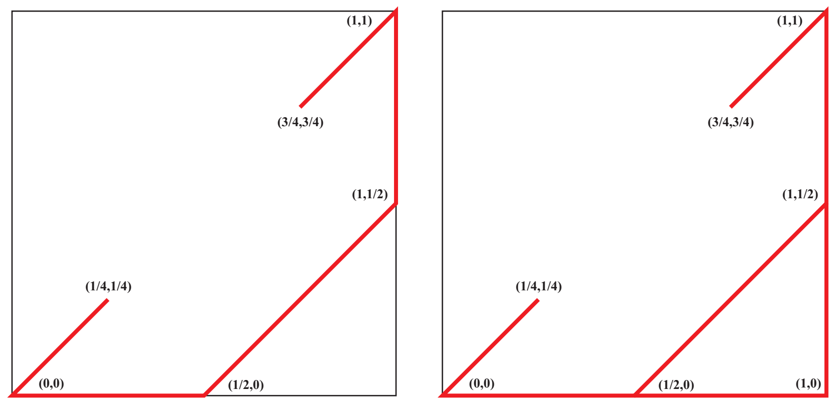

FIGURE 2. The graph of the bonding function f (left) and its composition with itself f^2 (right) in Example 4.2.

For the function f in Example 4.1, the graph of f^2 is not connected; $G(f^2) = ([0,1] \times \{1/2\}) \cup ([0,1] \times \{1\})$. The following example is a surjective upper semi-continuous function on $[0,1]$ having a graph that is an arc and all compositions have connected graphs but its inverse limit is not connected. This answers Question 5.1 of [1]. This example is adapted from an example due to Greenwood and Kennedy [5, Example 2.27, p. 45]. The proof that the inverse limit in Example 4.2 is not connected is much the same as that given in [5] but we include a proof here for completeness. Neither their original example nor its modification presented here has a connected inverse limit. Also, it may be of interest that if f is the function in Example 4.2, f^n for $n > 1$ is the original Greenwood and Kennedy function (pictured on the right as the graph of f^2 in Figure 2).

Example 4.2. *Let $f : [0,1] \nearrow [0,1]$ be the upper semi-continuous function whose graph consists of five straight line intervals, one from $(1/4,1/4)$ to $(0,0)$, one from $(0,0)$ to $(1/2,0)$, one from $(1/2,0)$ to $(1,1/2)$, one from $(1,1/2)$ to $(1,1)$, and one from $(1,1)$ to $(3/4,3/4)$. Then $G(f)$ is an arc and $G(f^n)$ is connected for each positive integer n but $\varprojlim \boldsymbol{f}$ is not connected.* (The graph of f is depicted on the left in Figure 2).

Proof. We show that G_3' is not connected by showing that $\boldsymbol{p} = (1/4, 1/4, 3/4, 3/4)$ is an isolated point of G_3' and apply Theorem 1.1. Let $U = (1/8,3/8) \times (1/8,3/8) \times (5/8,7/8) \times (5/8,7/8)$ and note that $\boldsymbol{p} \in U \cap G_3'$. Suppose $\boldsymbol{x} \in U \cap G_3'$. Because $x_1 \in (1/8,3/8)$, $x_1 \in f(x_2)$, and $x_2 \in (1/8,3/8)$, it follows that $x_2 \in (1/8,1/4]$. Because $x_2 \in (1/8,1/4]$, $x_2 \in f(x_3)$, and $x_3 \in (5/8,7/8)$, we see that $x_3 \in (5/8,3/4]$; because

$x_3 \in f(x_4)$ and $x_4 \in (5/8, 7/8)$ it follows that $x_4 = 3/4$. Thus, $x_3 = 3/4$ so $x_2 = 1/4$ and $x_1 = 1/4$ so $\boldsymbol{x} = \boldsymbol{p}$. Therefore, $\boldsymbol{p}$ is an isolated point of G_3' and so G_3' is not connected.

It is not difficult to see that $G(f^2)$ is the graph of the original Greenwood and Kennedy function and that, for $n > 2$, $G(f^n) = G(f^2)$. Thus, $G(f^n)$ is connected for each positive integer n. □

4.1. **Triods.** Continua that are inverse limits with set-valued functions having graphs that are arcs can contain triods. Such is the case in our next example, perhaps the simplest such example, where the inverse limit is a fan. Although this example has appeared in print many times [5, Example 2.13], we include it here not only because the graph is an arc and the inverse limit contains triods but also because it has some additional features that we note as motivation for some of our stated problems (including Problems 5.10 and 5.17). Other examples using graphs that are arcs include an inverse limit that contains a 2-cell [5, Example 5.3] as well as an infinite dimensional inverse limit [5, Example 2.3] and a one-dimensional inverse limit containing uncountably many mutually exclusive triods [5, Example 2.15]. Further such examples include [5, Example 3.11, pp. 57–58] and [7, Example 7.1].

Example 4.3. *Let* $f : [0,1] \nearrow [0,1]$ *be the upper semi-continuous function given by* $f(t) = \{0, t\}$ *for each* $t \in [0,1]$*. Then,* $G(f)$ *is an arc but* $\varprojlim \boldsymbol{f}$ *and* $\varprojlim \boldsymbol{f^{-1}}$ *are treelike continua that contain triods.*

Proof. Let $M = \varprojlim \boldsymbol{f}$ and $N = \varprojlim \boldsymbol{f^{-1}}$. The graph of f is clearly an arc. Because f^{-1} is interval valued, N is a continuum [5, Theorem 2.7, p. 18]. Thus, M is a continuum [5, Theorem 2.3, p. 16]. Both inverse limits are one-dimensional because neither bonding function contains both a vertical and a horizontal interval [13, Theorem 5.5, p. 1332]. Treelikeness for $\varprojlim \boldsymbol{f^{-1}}$ now follows from its trivial shape [3]. Because f^{-1} is continuum-valued, by Theorem 3.4, $\varprojlim \boldsymbol{f}$ has trivial shape, and, thus, $\varprojlim \boldsymbol{f}$ is also treelike.

Note that $G'(f,f)$ is the simple triod $\{\boldsymbol{x} \in [0,1]^3 \mid x_1 = x_2 = 0\} \cup \{\boldsymbol{x} \in [0,1]^3 \mid x_1 = 0 \text{ and } x_2 = x_3\} \cup \{\boldsymbol{x} \in [0,1]^3 \mid x_1 = x_2 = x_3\}$. Because $G(f)$ contains the graph of the identity on $[0,1]$, using Theorem 2.2 we see that M contains a triod. Because $G'(f,f)$ and $G'(f^{-1}, f^{-1})$ are homeomorphic and $G(f^{-1})$ contains the identity, N similarly contains a triod.

□

4.2. **Simple closed curves.** The author circulated a draft of this manuscript in early 2015 and one of the problems (Problem 5.12) we posed in it was quickly solved by Itzok Banič, Matevž Črepnjak, and Van Nall [2].

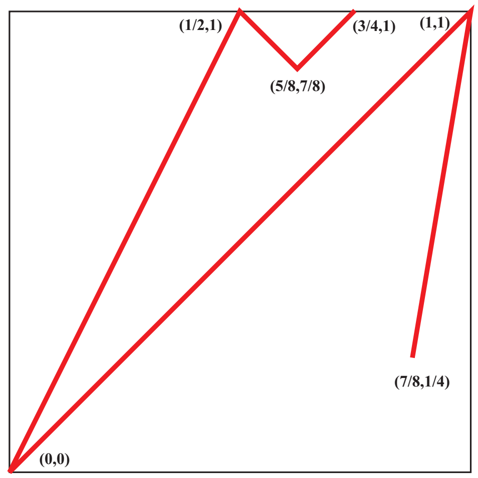

FIGURE 3. The graph of the bonding function f in Example 4.4.

We are keeping that problem in the problem section of this paper because they cite it. In our next example we modify their example slightly and prove that the inverse limit with this function also solves Problem 5.12. Because this graph is a subset of theirs, this also shows that their function produces an inverse limit that contains a simple closed curve.

Example 4.4. *Let* $f : [0,1] \nearrow [0,1]$ *be the upper semi-continuous function whose graph consists of five straight line intervals,* ℓ_1 *from* $(0,0)$ *to* $(1/2,1)$*,* ℓ_2 *from* $(1/2,1)$ *to* $(5/8,7/8)$*,* ℓ_3 *from* $(5/8,7/8)$ *to* $(3/4,1)$*,* ℓ_4 *from* $(0,0)$ *to* $(1,1)$*, and* ℓ_5 *from* $(7/8,1/4)$ *to* $(1,1)$*. Then,* $G(f)$ *is an arc but* $\varprojlim \boldsymbol{f}$ *is a one-dimensional nonconnected compactum that contains a simple closed curve.*

Proof. Let $M = \varprojlim \boldsymbol{f}$. Because f has zero-dimensional values, $\dim(M) \leq 1$, [13, Theorem 5.3, p. 1330]. Because M contains the arc $\{\boldsymbol{x} \in [0,1]^\infty \mid x_{i+1} = x_i \text{ for } i = 1,2,3,\dots\}$, $\dim(M) = 1$.

We show that $G'(f,f)$ contains a simple closed curve. Then, using the fact that $G(f)$ contains the graph of the identity on $[0,1]$, Theorem 2.2 allows us to conclude that M contains a simple closed curve. Note that $G(f)$ is the union of five homeomorphisms on intervals, $h_i, 1 \leq i \leq 5$, where for each i, the graph of h_i is the line ℓ_i. Let $\alpha_1 = G'(h_5,h_2)$; $\alpha_2 = G'(h_5,h_3)$; $\alpha_3 = G'(h_4,h_2)$; $\alpha_4 = G'(h_4,h_3)$. Then, $\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ is a simple closed curve lying in $G'(f,f)$.

Let $k_1 = h_1|[1/4, 1/2]$, $k_2 = h_5|[7/8, 11/12]$, $k_3 = h_5|[11/12, 15/16]$, and $k_4 = h_5|[15/16, 23/24]$. Let $\beta_1 = G'(k_1, k_2)$; $\beta_2 = G'(h_2, k_3)$; $\beta_3 = G'(h_3, k_4)$. Then, $\beta = \beta_1 \cup \beta_2 \cup \beta_3$ is an arc lying in G'_2 and $U = \{(x_1, x_2, x_3) \in [0,1]^3 \mid x_2 < x_1 \text{ and } x_3 > 3/4\}$ is an open set containing β that contains no point of $G'_2 - \beta$. So, G'_2 is not connected; thus, M is not connected. □

For reasons similar to those in the previous paragraph the example of Banič, Črepnjak, and Nall is not connected. A connected example could be of interest.

5. PROBLEMS

Assuming that the bonding functions in an inverse limit sequence on $[0, 1]$ have graphs that are arcs is a natural assumption given that the graphs of continuous functions from $[0, 1]$ into itself are arcs. This section includes some problems suggested by consideration of inverse limits with set-valued functions on $[0, 1]$ having graphs that are arcs. A number of these problems have been posed before in a more general context. On the other hand, many of the problems in this list that have not been previously posed elsewhere might well be raised in a more general setting. For this reason, in several problems we put the phrase 'whose graph is an arc' in parenthesis effectively posing two problems, one with and one without the parenthetical hypothesis. As with many questions involving inverse limits with set-valued function, the most useful answers to these problems are likely to be in terms involving properties of the bonding functions that are easy to verify. Certainly, this is not a definitive list of such problems, but the author hopes this can serve as a starting place for a number of successful investigations of inverse limits with set-valued functions having graphs that are arcs.

In light of Example 4.2 of this paper, our first problem seems natural.

Problem 5.1. *Suppose $f : [0, 1] \nearrow [0, 1]$ is a surjective upper semi-continuous set-valued function such that $G(f^n)$ is an arc for each positive integer n. Is $\varprojlim \boldsymbol{f}$ connected?*

The question of when an inverse limit with set-valued functions is connected has been posed and addressed often. Greenwood and Kennedy have characterized connectedness of inverse limits with set-valued functions on $[0, 1]$ in a paper to appear in Fundamenta Matematicae [4]. However, in the setting of inverse limit systems on $[0, 1]$ in which the bonding functions have graphs that are arcs, a characterization of connectedness (resp., a sufficient condition for connectedness) would be of interest, particularly if the condition is easy to check.

Problem 5.2. *Suppose* $\boldsymbol{f}$ *is a sequence such that* $f_i : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous function having a graph that is an arc for each positive integer* i*. Find necessary and sufficient conditions that* $\varprojlim \boldsymbol{f}$ *be connected.*

Problem 5.3. *Suppose* $\boldsymbol{f}$ *is a sequence such that* $f_i : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous function having a graph that is an arc for each positive integer* i*. Find sufficient conditions for* $\varprojlim \boldsymbol{f}$ *to be connected.*

The next two problems are the single bonding function versions of the previous two problems. These versions of those problems may be more tractable and would be of interest.

Problem 5.4. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find necessary and sufficient conditions that* $\varprojlim \boldsymbol{f}$ *be connected.*

Problem 5.5. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find sufficient conditions on the bonding function for* $\varprojlim \boldsymbol{f}$ *to be connected.*

It has long been known that a set-valued function on $[0,1]$ that is not a mapping can produce an arc as its inverse limit, [5, Example 3.1]. More recently, examples have been published showing that set-valued functions on $[0,1]$ can produce chainable continua that are not arcs ([6, Example 5.1] and [7, Example 7.1]) leading us to the following problem.

Problem 5.6. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find necessary and sufficient conditions that* $\varprojlim \boldsymbol{f}$ *be chainable.*

Problem 5.7. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find sufficient conditions on the bonding function for* $\varprojlim \boldsymbol{f}$ *to be chainable.*

Often chainability of an inverse limit with a set-valued function on $[0,1]$ fails due to the existence of a triod in the inverse limit. Example 4.3 shows that even if the graph of a single bonding function on $[0,1]$ is an arc, its inverse limit can contain a triod. See also Example 3.11 of [5, pp. 57–58] as cited earlier. Such examples prompt the next three problems.

Problem 5.8. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function (whose graph is an arc). Find sufficient conditions on the bonding function for $\varprojlim \boldsymbol{f}$ to be an atriodic continuum.*

Problem 5.9. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function (whose graph is an arc). Find necessary and sufficient conditions that $\varprojlim \boldsymbol{f}$ be an atriodic continuum.*

Problem 5.10. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function (whose graph is an arc). If $\varprojlim \boldsymbol{f}$ is atriodic, is $\varprojlim \boldsymbol{f^{-1}}$ atriodic?*

Problem 5.11. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function (whose graph is an arc). If $\varprojlim \boldsymbol{f}$ is an atriodic treelike continuum, is it chainable?*

Chainability would fail in an inverse limit on $[0,1]$ with set-valued functions if the inverse limit contains a simple closed curve. Our next problem appeared in an earlier draft of this article but it has subsequently been solved although we do not know of a connected example. The original solution was published by Banič, Črepnjak, and Nall in [2]. Example 4.4 above is a modification of their example that also settles the question.

Problem 5.12. (**Solved**) Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function whose graph is an arc. If $\varprojlim \boldsymbol{f}$ is one-dimensional, can it contain a simple closed curve? (Yes)

Problem 5.13. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function (whose graph is an arc). Find sufficient conditions on the bonding function for $\varprojlim \boldsymbol{f}$ to be an acyclic continuum.*

Problem 5.14. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function (whose graph is an arc). Find necessary and sufficient conditions that $\varprojlim \boldsymbol{f}$ be an acyclic continuum.*

Treelikeness appears to be closely tied to dimension in inverse limits on $[0,1]$ with set-valued functions. Examples show that an inverse limit on $[0,1]$ with bonding functions having graphs that are arcs can be infinite dimensional [5, Example 2.3, p. 19] or have any finite dimension [5, Chapter 5, pp. 70–73].

Problem 5.15. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function whose graph is an arc. Find sufficient conditions on the bonding function for $\varprojlim \boldsymbol{f}$ to be a treelike continuum.*

Problem 5.16. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find necessary and sufficient conditions that* $\varprojlim \boldsymbol{f}$ *be treelike.*

Problem 5.17. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function (whose graph is an arc) and* $\varprojlim \boldsymbol{f}$ *is a treelike continuum. Is* $\varprojlim \boldsymbol{f}^{-1}$ *treelike?*

On $[0,1]$, it is known that if the bonding function is interval-valued (resp., its inverse is interval-valued), then the inverse limit has trivial shape, [3] (resp., Theorem 3.4). Among one-dimensional continua treelikeness is characterized by the property of having trivial shape so the following three problem are of interest.

Problem 5.18. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function (whose graph is an arc). Find sufficient conditions that* $\varprojlim \boldsymbol{f}$ *have trivial shape.*

Problem 5.19. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find necessary and sufficient conditions that* $\varprojlim \boldsymbol{f}$ *be one-dimensional.*

Problem 5.20. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find sufficient conditions on the bonding function for* $\varprojlim \boldsymbol{f}$ *to be one-dimensional.*

Pertaining to Problem 5.19, it is known that if $f : [0,1] \nearrow [0,1]$ is an upper semi-continuous set-valued function such that $G(f)$ does not contain both a horizontal and a vertical interval then $\varprojlim \boldsymbol{f}$ is one-dimensional [13, Theorem 5.5, p. 1332]. However, there are examples of functions having a treelike inverse limit even though their graphs are arcs that contain both horizontal and vertical intervals [8, Example 5.3, pp. 628-629].

Inverse limits with upper semi-continuous bonding functions on $[0,1]$ having graphs that are arcs can be indecomposble [5, Example 3.9, p. 56].

Problem 5.21. *Suppose* $f : [0,1] \nearrow [0,1]$ *is a surjective upper semi-continuous set-valued function whose graph is an arc. Find sufficient conditions that* $\varprojlim \boldsymbol{f}$ *be indecomposable.*

Indecomposability in inverse limits with set-valued bonding functions appears to be tied to the full projection property (see [5, Section 3.5] for the definition and additional information). Problem 5.21 leads to our next problem.

Problem 5.22. *Suppose $f : [0,1] \nearrow [0,1]$ is a surjective upper semi-continuous set-valued function whose graph is an arc. Find sufficient conditions that $\varprojlim \boldsymbol{f}$ have the full projection property.*

Problem 5.23. *In each problem in this section, make the additional assumption that the graph of each function is the union of finitely many straight line intervals.*

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