A BRIEF HISTORICAL VIEW OF CONTINUUM THEORY

W. T. INGRAM

ABSTRACT. We explore a few topics in continuum theory from their roots. Specifically, we examine the evolution of the definition of *continuum* and then restrict most of our attention to one-dimensional continua. Particular attention is paid to indecomposable continua, the fixed point property, hereditary equivalent continua, homogeneous continua, chainable continua and span of continua. In this paper, we give an inverse limit description of an indecomposable circle-like continuum that is homeomorphic to the first example of an indecomposable continuum given by L. E. J. Brouwer in 1910.

1. INTRODUCTION

This article[1] is not intended to be a historian's treatment of the history of continuum theory for I am not a historian. Rather, I will trace some of the ideas in continuum theory back near, if not actually to, their genesis. The writers of the late 1800s, just as is the case with writers today, do not appear always to have been fully aware of the work of some of their contemporaries. I will give what may be an example of this in the first paragraph of Section 2. As a consequence, I cannot be certain that some things that appear to be "first" were indeed not without some precedent. Some of the sources I read in the preparation of the talk and which I quote in the present article include [17], [18], [26], [28], [34], [37], [40], [44], [45]. Additional pertinent articles, some of which are not actually cited within this paper, may be found in the references.

I make no claim, nor do I attempt, to cover every aspect of continuum theory in this paper. The major interests of the experts

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[1]This paper is an outgrowth of a talk on the history of continuum theory before 1990 given at the University of Alabama at Birmingham, February 22, 2003, at a conference marking E. D. Tymchatyn's sixtieth birthday.

in continuum theory will not all be addressed. Indeed, not even all of my own personal interests have found their way into these pages. I apologize now if I omit saying something about a particular reader's favorite niche of continuum theory. My task for my talk was to discuss the pre-1990 era, so I do not really delve into many of the things I have enjoyed working on over the past several years.

I will limit these remarks to one-dimensional continua, the so called *curves*. There are several reasons for this:

1. this is where I have done most of my own work,
2. there is plenty of material and
3. continuum theory just may have been born out of attempts to understand curves.

2. Early developments: 1883–1922

Organizing the talk from which this article grew presented a major problem. In my talk I chose a time-line as the principal tool in its format and I adhere to that in this article as well. Sometimes, when it is appropriate, I will break strict adherence to the timeline to trace an idea further into its development. J. H. Manheim, [37, p. *xii*], says in his preface, "A study such as this must begin somewhere ..." . So, we begin in 1883 with G. Cantor's definition of a continuum [11]: a perfect set in E^n such that for each two points a, b of it and for each $\varepsilon > 0$ there corresponds a finite system of points $p_0 = a, p_1, p_2, \ldots, p_n = b$ of it such that $|p_i - p_{i+1}| < \varepsilon$, $0 \leq i < n$. In the presence of compactness Cantor's finite system of points condition is equivalent to the modern day meaning of connected in metric spaces. In [11], Cantor provides an example (the classic middle-thirds Cantor set) to show how far perfect alone is from capturing the notion of a curve. So, it appears that the Cantor set was invented to justify Cantor's definition of a "continuum." Interestingly, eight years earlier, H. J. S. Smith gave an example of a Cantor set plus countably many isolated points in a paper on integration theory [42], a paper perhaps not known to Cantor.

Today, we recognize the salient features of Cantor's definition to be that the set be closed and connected—the definition of continuum used, for example, by R. L. Moore [40]. The definition of connected evolved until it was divorced from the metric setting and

given its modern meaning independently by N. Lennes and F. Riesz, [45].

Most continuum theorists now define a continuum to be a compact, connected subset of a metric space though some take a somewhat broader view and study compact, connected Hausdorff spaces. The study of compactness can be traced to E. Borel's thesis in 1894 where he proved that countable open covers of the interval contain finite subcovers [18]. E. Lindelöf subsequently showed the assumption of countability is unnecessary [18]. The term compact was introduced by M. Fréchet in 1904 to describe the property that every sequence has a convergent subsequence [18]. Compact in the modern sense was introduced by P. Alexandroff and P. Urysohn in 1923 [18].

Cantor remarked (see [45]) that his definition of continuum did not restrict dimension. This indicates that his intent was not to define the linear continuum of the real numbers. Cantor's definition notwithstanding and, although it is pure speculation on my part, I believe a lot of the motivation of the early work in continuum theory was to "define" curves (probably things they could draw) by their properties. The definition of small inductive dimension that evolved ruled out the disk as a "curve" but then gave rise to the consideration of the nature of boundaries of small open sets and probably led to the notions of dendrite, regular curve, rational curve, etc.

To make advances in real and complex analysis, it became imperative to gain a better understanding of the topology of Euclidean spaces. Thus, an early and important result was the Jordan Curve Theorem: a simple closed curve in the plane separates the plane into two mutually exclusive open sets such that it is the boundary of each of them. C. Jordan stated this in 1887 but the first rigorous proof was given in 1905 by O. Veblen [17]. Beginning around 1904, A. Schoenflies published several papers resulting from his studies of the plane. He had numerous results relating a continuous curve to its complement in the plane (including accessibility theorems). Moore states in the Appendix to [40] that "...Schoenflies has not received sufficient credit for all of the contributions which he made to point set theory." By sometimes relying too heavily on intuition, Schoenflies also made mistakes. One of these was that a curve that is the common boundary of two mutually exclusive open sets in the

plane is decomposable (i. e., the union of two of its proper subcontinua). L. E. J. Brouwer demonstrated this to be false in 1910 and described the first indecomposable continuum [9] (a continuum is *indecomposable* if it is not decomposable).

J. Kennedy [26] has given us a description of Brouwer's example. In the following, I give an indication of a proof that his example is homeomorphic to an inverse limit on circles using a single bonding map. This relies heavily on Kennedy's description. In order to save the reader from having to have her paper in hand to read this one, I have reproduced several of her pictures and have quoted extensively from her description of the example. For a more thorough description of the example, the reader is referred to Kennedy's article.

To obtain Brouwer's example, one begins with a rectangular 2-cell, R_0, in the plane. From this 2-cell one removes an open set D and an open set R_1 along with an open interval lying in the common boundaries of R_0 and R_1. This leaves a closed topological annulus, A_1, in the plane that one may chain with a circular chain $\mathcal{C}_1$ whose first link contains the lower left-hand corner of this annulus and proceeds around the annulus in a clockwise direction.
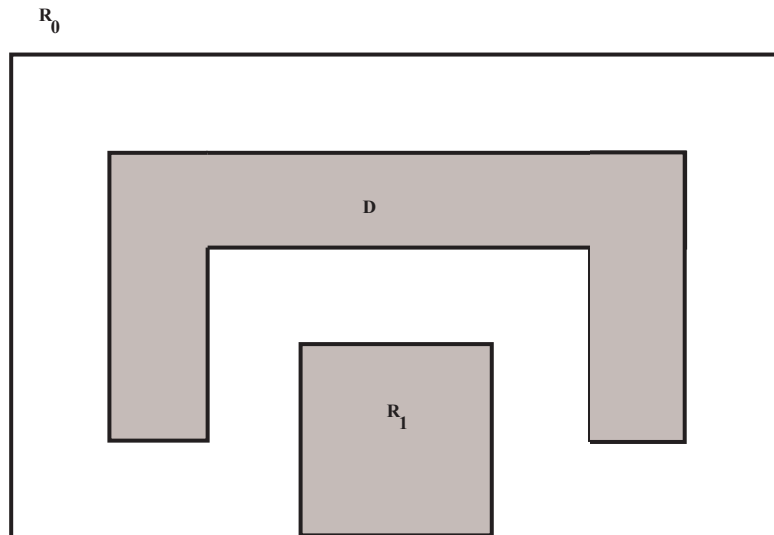

Figure 1

The second stage of the construction involves removal of two open sets R_2 and R_3 along with appropriate parts of their boundaries producing an annulus, A_2. Notice in Figure 2 that R_2 and R_3 run virtually "parallel" around the upper part of A_1. When the annulus A_2 is chained with a circular chain C_2 refining C_1 and whose first link contains the lower left-hand corner of this annulus and lies in the first link of C_1, C_2 circles through C_1 one and one-half times and then doubles back effecting circling once.
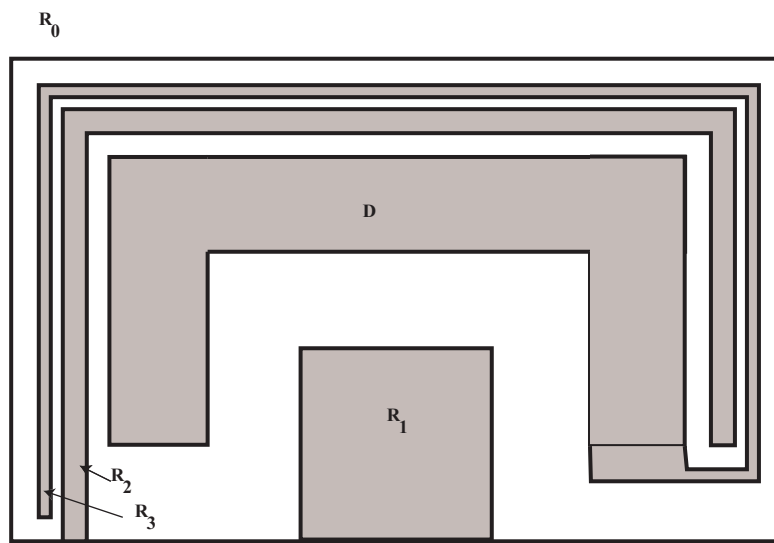

Figure 2

The third stage of the construction involves removal to two more open sets R_4 and R_5 along with appropriate parts of their boundaries producing an annulus A_3. Notice in Figure 3 that these two open sets also run virtually parallel through the first half of the annulus A_2. When A_3 is chained with a circular chain C_3 refining C_2 and whose first link contains the lower left-hand corner of this annulus and lies in the first link of C_2, C_3 circles through C_2 one and one-half times and then doubles back just as before.
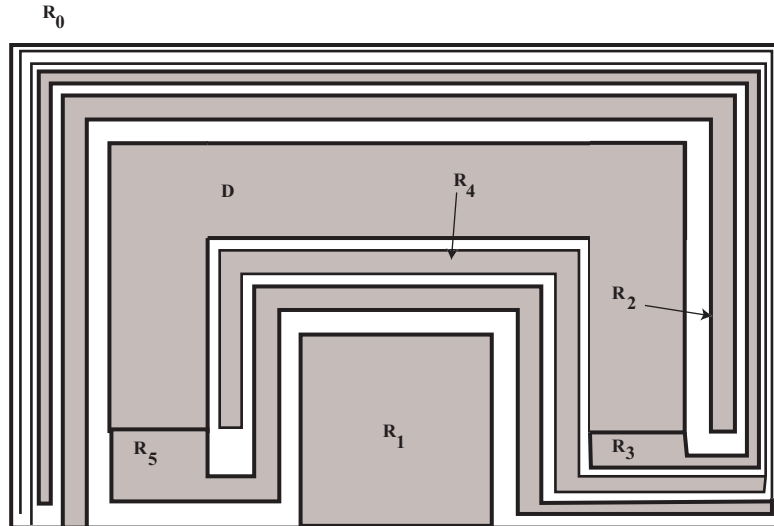
Figure 3

Continuing this process, we see that the Brouwer continuum is the intersection of this sequence of annuli, $A_1, A_2, A_3, \ldots$ and that it can be chained with a sequence of circular chains each succeeding term of which circles straight through its preceding term one and one-half times and doubles straight back.

Using standard techniques such at those used in [24], one can show that such a circle-like continuum is homeomorphic to an inverse limit on circles using a single bonding mapping f schematically illustrated in Figure 4. The map fixes $(1, 0)$, takes the top of the circle around the circle once with $(-1, 0)$ being thrown to $(1, 0)$. It takes the bottom half of the circle half-way around traversing the top and then folding back. Analytically, if S^1 denotes the unit circle in the complex plane, the map $f : S^1 \twoheadrightarrow S^1$ is given by

$$f(z) = \begin{cases} z^2 & \text{if } 0 \le \text{Arg}(z) \le 3\pi/2, \\ z^{-2} & \text{if } 3\pi/2 \le \text{Arg}(z) \le 2\pi. \end{cases}$$

That $\varprojlim\{S^1, f\}$ is indecomposable may easily be seen by observing that there are three points of S^1, $(1, 0), (-1, 0)$ and $(0, -1)$ such that if α is an arc lying in S^1 and containing two of these three points then $(f \circ f)(\alpha) = S^1$. By Kuykendall's theorem [29, Theorem 2, p. 267], this is sufficient that $\varprojlim\{S^1, f \circ f\}$ (and, therefore,

$\varprojlim\{S^1, f\})$ is indecomposable. However it is easy to argue directly that the existence of the three points $(1,0), (-1,0)$ and $(0,-1)$ of S^1 is sufficient for indecomposability of the inverse limit. We provide an argument in the next paragraph.

Let $M = \varprojlim\{S^1, f\}$ and assume H and K are proper subcontinua of M such that $M = H \cup K$. Let j be an integer such that if $n \geq j$ then $\pi_n(H) \neq S^1$ and $\pi_n(K) \neq S^1$. Then, since $\pi_{j+2}(H)$ and $\pi_{j+2}(K)$ are two subarcs of S^1 whose union is S^1, one of them, say $\pi_{j+2}(H)$, contains two of $(1,0), (-1,0)$ and $(0,-1)$. But, $\pi_j(H) = (f \circ f \circ \pi_{j+2})(H) = S^1$, contradicting the choice of j.
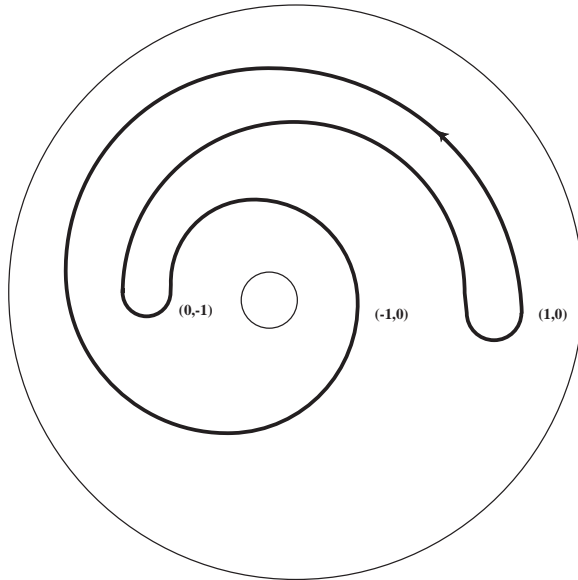

Figure 4

Z. Janiszewski [25], in 1911, inspired by Brouwer's example described a simplification of Brouwer's example that does not separate the plane; in 1922 B. Knaster gave us through C. Kuratowski, the familiar geometric description [28, p. 205, Fig. 4] of the Brouwer-Janiszewski-Knaster continuum (see Figure 5), a continuum that also can arise in the construction of the Smale horseshoe, [41]. In 1922, Knaster described an hereditarily indecomposable continuum

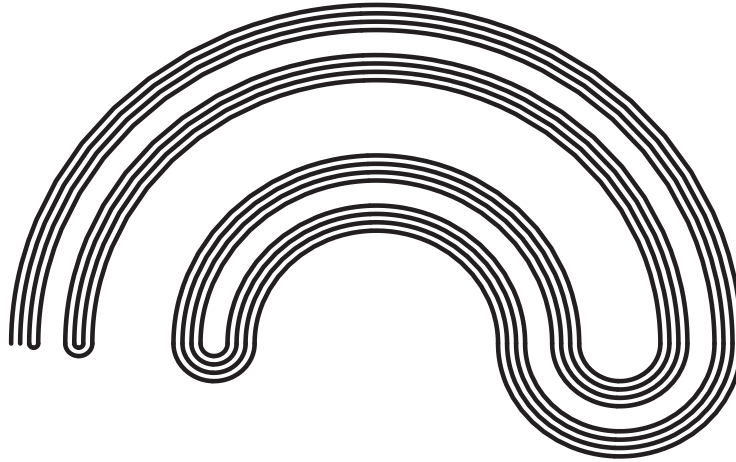(a continuum is *hereditarily indecomposable* provided each of its subcontinua is indecomposable) [27].

Figure 5

In 1912, Brouwer [10] influenced continuum theory in another way. He proved that the n-cell in Euclidean n-dimensional space has the fixed point property (although some think this may partially date back to P. Bohl [8] in 1904 for differentiable maps). This result has given rise to one of the famous (and as yet unsolved) problems of continuum theory: Does every non-separating plane continuum have the fixed point property? W. L. Ayres [2] appears to have first asked this question in print. In 1951 O. H. Hamilton showed that chainable continua (see the next paragraph for the definition) have the fixed point property [21]. Also in 1951, R H Bing [6] asked if tree-like continua have the fixed point property. Actually, in his 1951 paper Bing asked if *planar* tree-like continua have the fixed point property, but in 1969 in [7] the reference to the plane had disappeared. In 1979, Bellamy [3] provided us with an example of a tree-like continuum without the fixed point property; the question of whether every non-separating plane continuum has the fixed point property remains open.

In 1916, Moore [39] proved that in a locally connected, separable metric space, connected open sets are arcwise connected (Moore actually proved this in a more general setting than metric). His

method of proof spawned the study of chainable continua. A continuum M is *chainable* if for each $\varepsilon > 0$ there is a finite sequence of open sets D_1, D_2, ..., D_n covering M such that $D_i \cap D_j \neq \emptyset$ if and only if $|i - j| \leq 1$ and diam$D_i < \varepsilon$ for $1 \leq i \leq n$. Chainability of a continuum is equivalent to its being homeomorphic to an inverse limit on intervals.

3. Fundamenta Mathematicae, 1920–present

A very important date in the history of continuum theory is 1920—the year that marks the inaugural issue of the Polish Journal, Fundamenta Mathematicae. One could spend pages extolling the influences of this journal on the development of continuum theory. I will mention only a few.

In the very first issue, W. Sierpiński introduced the notion of homogeneity (a continuum is *homogeneous* provided for each two points p and q of it there is a homeomorphism h of the continuum onto itself such that $h(p) = q$) and later in Volume I, Knaster and Kuratowski asked whether each homogeneous plane continuum is a simple closed curve. In 1924, S. Mazurkiewicz [33] proved each locally connected, homogeneous plane continuum is a simple closed curve.

In Volume II of Fundamenta, Mazurkiewicz [32] asked if the arc is the only finite dimensional continuum homeomorphic to each of its non-degenerate subcontinua. Knaster's hereditarily indecomposable continuum (mentioned in the previous section) appeared in Volume III.

4. The pseudo-arc, 1948

Many important discoveries were made in the '20s, '30s and early '40s, but in the interest of getting to our lifetimes, let me jump to a watershed event in 1948. Just in passing, I will mention that in 1936 S. Eilenberg [19] demonstrated ways to determine topological information about a continuum from properties of the space of mappings of the continuum to the circle; in 1937, H. Freudenthal [20] described solenoids (studied earlier and shown by D. van Dantzig to be homogeneous in 1930) in terms of inverse limits. In 1948, E. E. Moise [38] gave a surprising answer to Mazurkiewicz' question of 1921—the arc is not the only hereditarily equivalent

continuum. Moise dubbed his example the pseudo-arc. It is an indecomposable chainable continuum, so being hereditarily equivalent, it is hereditarily indecomposable. That same year (1948), Bing [4] showed that the pseudo-arc answers the question of Knaster and Kuratowski in the negative—the pseudo-arc is another homogeneous plane continuum. In 1951, Bing [5] showed that each two hereditarily indecomposable chainable continua are homeomorphic. Thus, since his example is chainable, Knaster had actually built the pseudo-arc in 1922 when he constructed the first hereditarily indecomposable continuum. Much more information on the pseudo-arc can be found in the work of W. Lewis. In particular the reader is referred to [31].

5. The late fifties and the sixties

In 1959, R. D. Anderson and G. Choquet [1] constructed a non-separating plane continuum no two of whose non-degenerate subcontinua are homeomorphic. This paper showed the potential of inverse limits to build complicated objects out of simple ones. In 1967, H. Cook [13] adapted the Anderson-Choquet technique to construct a continuum whose only non-constant continuous self-transformation is the identity.

Two events significant in this story occurred in 1960. G. Henderson [22] proved that the arc is the only decomposable hereditarily equivalent continuum (i. e., homeomorphic to each of its non-degenerate subcontinua); J. H. Case and R. E. Chamberlin [12] characterized tree-likeness of continua. The Case-Chamberlin characterization is: a continuum is tree-like if and only if it is one-dimensional and every mapping of the continuum to a one-dimensional polyhedron is inessential. Due to the limited scope of this article, I will not delve more deeply into the study of the space of mappings of a continuum into a polyhedron such as the circle or the figure eight although this is a rich aspect of continuum theory begun, as mentioned earlier, by Eilenberg. Indeed, I mention the Case-Chamberlin result mainly because ten years later in 1970 Cook used it in showing that dendroids are tree-like [14] and that hereditarily equivalent continua are tree-like [15]. Cook's result, along with Henderson's, represents the current state of our knowledge on the Mazurkiewicz problem on hereditarily equivalent

continua, although L. Oversteegen and E. D. Tymchatyn [36] have shown that an hereditarily equivalent continuum in the plane has symmetric span zero. For more information on span zero including the definition see the next paragraph. It is not known if hereditarily equivalent continua must have span zero.

A question that has been a sort of "guiding star" for my research over my career has been: What internal properties of continua characterize chainability? In 1964, A. Lelek [30] defined a property of chainable continua which may characterize chainability—span zero. A continuum has *span zero* provided every subcontinuum of the product of the continuum with itself having one projection lying in the other must intersect the diagonal (*symmetric span zero* requires that every subcontinuum of the product having both projections the same must intersect the diagonal). Span zero gave me precisely the tool to show in 1972 that there exists an atriodic tree-like continuum that is not chainable [23]. If you look over Tymchatyn's papers, you will see he has had a keen interest in whether continua with span zero must be chainable. If they are, then a result of Oversteegen and Tymchatyn in 1982 [35] yields that we know all of the homogeneous plane continua.

6. The Houston Seminar and the Houston Problem Book

In 1971, the seminar at Houston began–the seminar that led to the Houston Problem Book [16]. Lelek made sure that we had a permanent record of the seminar by purchasing the books into which the seminar notes were entered. These books led directly to the creation of the Houston Problem Book. Through the cooperation of my institution, the University of Missouri - Rolla, the notebooks from that seminar have been scanned and made available on-line through the University of Missouri Library archives at http://digital.library.umsystem.edu/ebind/ebindsamples.html.

Tymchatyn's work was often featured in this seminar. The very first problem raised at the very first meeting of the seminar was solved by Tymchatyn. Indeed, the first three problems are problems of Tymchatyn who has solved Problems 1 and 3 while Problem 2 was still open in 1995. Moreover, the paper that was presented in the second and third meetings of the seminar was one written

by Tymchatyn, *Continua whose connected subsets are arcwise connected* [43].

   Along with Lelek, Cook gets a lot of credit for the existence of Houston Problem Book. He wrote a computer program designed specifically to create a database from which the first edition of the problem book was produced. I was responsible for rescuing the files from the University of Houston mainframe computer and getting them onto a desktop computer for the first time. The text files were subsequently formatted as AMS TEXfiles by a graduate student at Houston, R. Henderson. It was these TEXfiles that we updated to produce the paper containing the published edition of the problem book, [16].

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MISSOURI–ROLLA, ROLLA, MO 65409–0020

  E-mail address: ingram@umr.edu