Inverse Limits and Dynamical Systems

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Abstract

In this paper we state some problems that arise in the study of inverse limits. Many of the problems come from research in inverse limits inspired by considerations from dynamical systems. Areas from which the problems are chosen include chainable continua, plane embeddings, inverse limits on $[0, 1]$, the Property of Kelley, and inverse limits using upper semi-continuous bonding functions. Problems related to recent developments in applications of inverse limits to models arising from economics constitute the final section of the paper.

Key words: inverse limit, chainable, tent map, Property of Kelley, upper semi-continuous function, plane embedding, backward dynamics
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1 Introduction

Throughout this article, we use the term *continuum* to mean a compact, connected subset of a metric space; by a *mapping* we mean a continuous function. A continuum is *decomposable* provided it is the union of two of its proper subcontinua; a continuum is *indecomposble* if it is not decomposable. A continuum is *hereditarily decomposable* if each of its subcontinua is decomposable.

If $X_1, X_2, X_3, \ldots$ is a sequence of metric spaces and $f_1, f_2, f_3, \ldots$ is a sequence of mappings such that $f_i : X_{i+1} \to X_i$ for $i = 1, 2, 3, \ldots$, by the *inverse limit* of the inverse limit sequence $\{X_i, f_i\}$ is meant the subset of the product space $\prod_{i>0} X_i$ that contains the point $(x_1, x_2, x_3, \ldots)$ if and only if $f_i(x_{i+1}) = x_i$ for each positive integer i. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is denoted by $\varprojlim \{X_i, f_i\}$. For convenience of notation, we will use boldface characters to denote sequences. Thus, if $s_1, s_2, s_3, \ldots$ is a sequence, we denote this sequence by $\mathbf{s}$. By this convention, the point $(x_1, x_2, x_3, \ldots)$ of an inverse limit space will also be denoted by $\mathbf{x}$, the sequence of factor spaces by $\mathbf{X}$ and the sequence of bonding maps by $\mathbf{f}$. For brevity, we will denote the inverse limit space by $\varprojlim \mathbf{f}$.

A problem set is invariably personal and reflects the interests of the compiler of the set. So it is with this collection of problems. Because of recent developments in the use of inverse limits in certain kinds of models in economics, in Section 7 we include some problems arising from this although we have not personally contributed anything to these applications. Instead we rely on some who have made contributions for problems that reflect the current state of this research.

2 Characterization of Chainability

Although it is not the original definition of chainability we take as our definition that a continuum is *chainable* to be that the continuum is homeomorphic to an inverse limit on intervals; a continuum is *tree-like* provided it is homeomorphic to an inverse limit on trees. A continuum is *unicoherent* provided it is true that if it is the union of two subcontinua H and K then $H \cap K$ is connected; a continuum is *hereditarily unicoherent* provided every subcontinuum of it is unicoherent. A continuum M is a *triod* provided there is a subcontinuum H of M so that $M - H$ has at least three components; a continuum is *atriodic* provided it contains no triod. It is immediate that chainable continua are tree-like. It is well known that chainable continua are atriodic and tree-like continua are hereditarily unicoherent.

Several characterizations of chainability of a continuum exist. These include (1) (the original definition) for each $\varepsilon > 0$ there is a finite collection of open sets $C_1, C_2, \ldots, C_n$ covering M such that $\mathrm{diam}(C_i) < \varepsilon$ for $1 \leq i \leq n$ and $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$ and (2) for each positive number ε there is a map f_ε of the continuum to $[0,1]$ such that if t is in $[0,1]$ then the diameter of $f_\varepsilon^{-1}(f_\varepsilon(t))$ is less than ε. Notably missing is a characterization involving a list of internal topological properties of the continuum. For example, in case the continuum is hereditarily decomposable, R H Bing [3, Theorem 11] proved that the continuum is chainable if and only if it is atriodic and hereditarily unicoherent. This characterization for hereditarily decomposable continua is satisfying in that it is given in terms of "internal" topological properties of the continuum.

Problem 1 *Characterize chainability of a continuum in terms of internal topological properties of the continuum.*

J. B. Fugate [9] extended Bing's result from the class of hereditarily decomposable continua to the class of those continua having the property that every indecomposable subcontinuum is chainable. Thus, Problem 1 may be solved by characterizing chainability of indecomposable continua. Case and Chamberlin [6] gave a characterization of tree-like continua as those one-dimensional continua for which every mapping to a one-dimensional polyhedron is inessential (i.e., homotopic to a constant map). J. Krasinkiewicz later proved that a one-dimensional continuum is tree-like if and only if every mapping of it to a figure-8 (the union of two circles with a one-point intersection) is inessential [26]. Although these characterizations of tree-likeness do not involve "internal" topological properties, it would still be of significant interest to characterize chainability among tree-like continua. Since tree-like continua are hereditarily unicoherent, atriodicity is a natural candidate for one of the properties on a list of characterizing properties. That atriodicity alone is not sufficient was shown in [13].

One significant attempt at characterizing chainability involves the notion of the span of a continuum. If M is a continuum, the *span* of M is the least upper bound of $\{\varepsilon \geq 0 \mid$ there is a subcontinuum C of $M \times M$ such that $p_1(C) = p_2(C)$ and $\mathrm{dist}(x,y) \geq \varepsilon$ for all (x,y) in $C\}$ (p_1 and p_2 denote the projections of $M \times M$ into its factors). The following problem on span remains open even though it was featured [8] in the first volume of *Open Problems in Topology.*

Problem 2 *If the span of a continuum is 0, is M chainable?*

A. Lelek intoduced span in [28] and proved that chainable continua have span 0. Although span 0 is a topological property, in some real sense it is not "internal". Consequently, if one were to settle Problem 2 in the affirmative, the nature of the definition of span would, in this author's opinion, leave work to be done on Problem 1. That said, Problem 2 is significant in its own right and not only because it has become an "old" problem. For instance, a positive solution would tell us that we know all of the homogeneous plane continua [34].

3 Plane Embedding

In thinking about Problem 1 and in light of the Case-Chamberlin theorem [6] characterizing tree-likeness, the author began a quest to settle the question whether atriodic tree-like continua are chainable. That investigation led to an example of an atriodic tree-like continuum that is not chainable [13]. Span turned out to be just the tool needed to show that the example obtained is not chainable.

However, span was not the first tool the author tried to use. In fact, two other properties of chainable continua first came to mind: planarity and the fixed point property. Bing showed that chainable continua can be embedded in the plane [3] and O. H. Hamilton showed that chainable continua have the fixed point property [11]. The author chose to try to employ Bing's result and construct an atriodic tree-like continuum that cannot be embedded in the plane. This leads to our next problem.

Problem 3 *Characterize those tree-like continua that can be embedded in the plane.*

The reader interested in this problem should be familiar with Brian Raines work [35] on local planarity of inverse limits of graphs.

Of course, tree-like continua that cannot be embedded in the plane are well known. Arguably the simplest of these may be one given by Bing [3]. This example consists of a ray with remainder a simple triod together with an arc that intersects the union of the ray and the triod only at the junction point of the triod. A map of the 4-od that produces in its inverse limit a simple triod and a ray having the simple triod as a remainder is shown in Figure 1. Bing's example is obtained by attaching an arc to this inverse limit at the point $(O, O, O, \ldots)$ and otherwise misses the inverse limit.

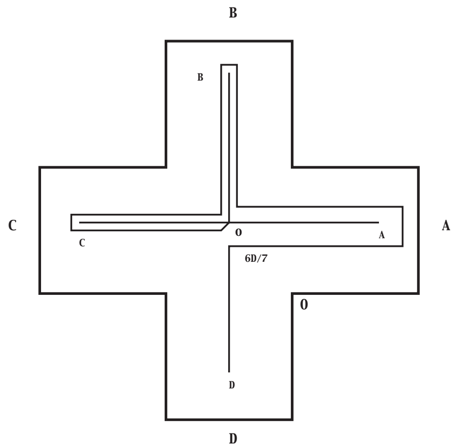

Figure 1

Finding an atriodic example presents somewhat more of a challenge. Although other non-planar atriodic tree-like continua are known, an example may be constructed in the following way. Let M be the atriodic tree-like continuum with positive symmetric span that the author constructed in [13] and let C be the product of M with a Cantor set. A construction of Laidacker [27] produces an atriodic tree-like continuum M' that contains C. Dušan Repovš, Arkadij B. Skopenkov, and Evgenij V. Ščepin prove in [36] that the plane does not contain uncountably many mutually exclusive tree-like continua with positive symmetric span so the continuum M' is non-planar.

In order to tackle Problem 3, it would be helpful to have some examples of planar continua that "look" like they might be non-planar as well as some simpler examples of non-planar atriodic tree-like continua to study. The remainder of this section is devoted to some examples and possible examples.

The author's first attempts to construct an atriodic tree-like continuum that cannot be embedded in the plane failed (as have many subsequent attempts). We briefly describe an early attempt. The picture in Figure 2 is a schematic drawing of a mapping f of a simple triod $T = [OA] \cup [OB] \cup [OC]$ onto itself. The action of the function is to take the first half of $[OA]$ onto $[AO]$ with $f(O) = A$ and the second half of $[OA]$ onto $[OB]$ with $f(A) = B$; f takes the first third of $[OB]$ onto $[AO]$, the next sixth onto the first half of $[OC]$, the next sixth folds back to O and the final third is taken onto $[OB]$; f takes the first third of $[OC]$ onto $[AO]$, the next third half way out $[OB]$ and back, and the final third onto $[OC]$. The resulting inverse limit is atriodic, but it is a chainable continuum because $f \circ f$ factors through $[0, 1]$ (i.e., there are maps $g : T \to [0, 1]$ and $h : [0, 1] \to T$ so that $f = h \circ g$). Though the schematic of f cannot be drawn in the plane, $\varprojlim \mathbf{f}$ being chainable is planar.

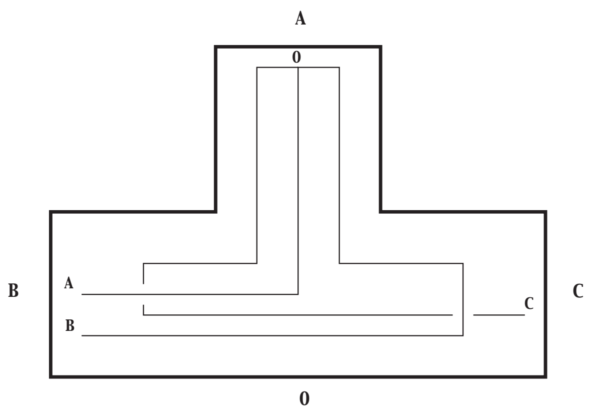

Figure 2

An alternative to Bing's non-planar tree-like continuum mentioned above is the following. Take a continuum consisting of two mutually exclusive rays each having the same simple triod as remainder but the rays wind around the triod in opposite directions. The resulting tree-like continuum is non-planar. This observation suggests the following way possibly to construct a non-planar atriodic tree-like continuum. The continuum is an inverse limit on a simple 5-od, $[OA] \cup [OB] \cup [OC] \cup [OD] \cup [OE]$. Restricted to the triod $[OA] \cup [OB] \cup [OC]$ our 5-od map is just the triod map that the author used in [13] to obtain an atriodic tree-like continuum that is not chainable. We use the other two arms of the 5-od to obtain rays that wind in "opposite" directions onto that example. We provide a schematic diagram of the map in Figure 3. The author does not know if the resulting inverse limit is non-planar.

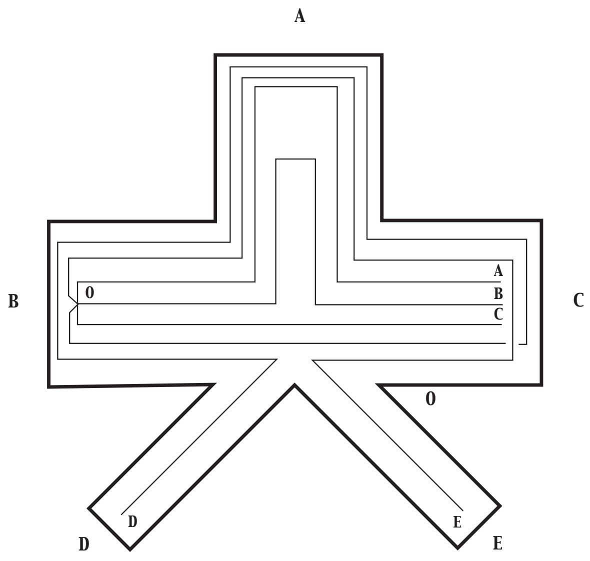

Figure 3

There is a somewhat simpler possibility that results from an inverse limit on 4-ods. The author does not know if the resulting inverse limit space is non-planar. The bonding map f (shown in a schematic in Figure 4) has the interesting feature that, although it can be "drawn in the plane", f^2 cannot be "drawn in the plane". This appears to be caused by a twist of the arms of the 4-od imposed by the bonding map. Unfortunately, as our second example shows (see Figure 2), not being able to "draw" a schematic of the bonding map in the plane does not guarantee that the inverse limit is non-planar.

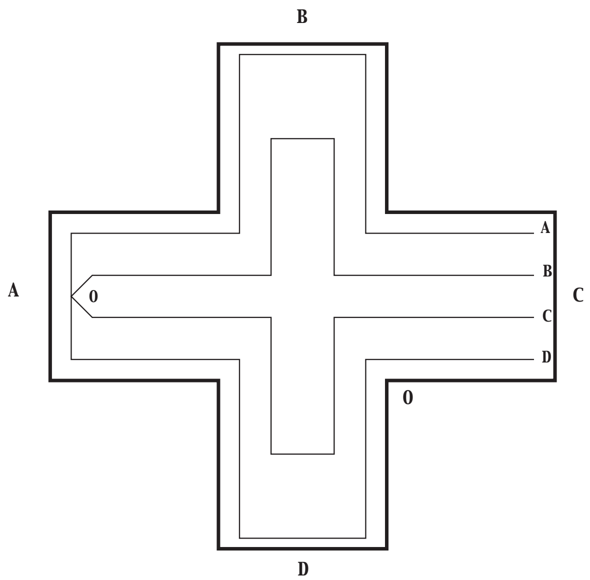

Figure 4

4 Inverse Limits on $[0,1]$

Considerable interaction between dynamacists and continuum theorists has occurred in the past fifteen or twenty years. Inverse limits appeal to dynamacists in part because they allow one to transform the study of a dynamical system consisting of a space and a mapping of that space into itself into the study of a (perhaps more complicated) space, the inverse limit, and a homeomorphism, the shift, of that space into itself. Considerations in dynamics have led to extensive investigations of parameterized families of maps. Many of these are maps of $[0,1]$ into itself and include the *logistic* family and the *tent* family. Interest in these families also rekindled the author's interest in inverse limits on $[0,1]$ using a constant sequence of bonding maps in which that bonding map is chosen from one of those two families or from one of several other families of piecewise linear maps including the families f_t for $0 \le t \le 1$, g_t for $0 \le t \le 1$, f_{ab} (also denoted g_{bc} by the author and others) where both parameters come from $[0,1]$, and the family of permutation maps. In this article we provide definitions only for the tent family (in the next paragraph) and the permutation maps (in the next section). For definitions of the families not discussed further in this article and information on some of the inverse limits generated by these families see [15], [16], [17], [21]. With one exception these are families of unimodal maps, a class of maps of special interest in dynamics. Permutation maps are Markov maps, a class also of interest in dynamics. A map is *monotone* provided its point inverses are connected; a map $f:[0,1]\to[0,1]$ is *unimodal* provided f is not monotone and there is a point c, $0<c<1$, such that $f|[0,c]$ and $f|[c,1]$ are monotone. A map $f:[0,1]\to[0,1]$ is *Markov* provided there is a finite subset $\{x_1=0, x_2, \ldots, x_n=1\}$ with $x_i < x_{i+1}$ and $f|[x_i, x_{i+1}]$ is monotone for $1 \le i < n$.

Tent maps are unimodal maps of $[0,1]$ constructed as follows. Choose a number s from $[0,1]$ and let $f_s:[0,1]\to[0,1]$ be the piecewise linear map that passes through $(0,0)$, $(1/2,s)$, and $(1,0)$. (Specifically, f_s is given by $f_s(x)=2sx$ for $0\le x\le 1/2$ and $f_s(x)=(2-2s)x$ for $1/2\le x\le 1$.)

One problem involving the tent family has sparked considerable interest and has given rise to a large number of partial results.

Problem 4 *If f_s and f_t are tent maps with $\varprojlim \mathbf{f_s}$ and $\varprojlim \mathbf{f_t}$ homeomorphic, is $s=t$?*

This has been settled in a number of cases including Lois Kailhofer's proof for maps that have periodic critical points [23]. Štimac has announced a positive solution if the maps have preperiodic critical points.

The collection of inverse limits arising from the tent family is rich in its variety. Barge, Brucks, and Diamond have shown that there are uncountably many parameter values at which the inverse limit is so complicated that it contains a copy of every continuum arising as an inverse limit space from a tent family core (see the next paragraph) [2]. In spite of the presence of complicated topology at some parameter values, progress has been made on Problem 4 when the orbit of the critical point is infinite. B. Raines has begun a systematic study of these inverse limits and has made some significant progress for certain parameter values. The author acknowledges private correspondence with Professor Raines that provided some of the problems in this section as well as some of the information on the literature related to these problems.

If f_s is a tent map $\varprojlim \mathbf{f_s}$ is the closure of a topological ray. Except for $s = 1$ the inverse limit is a decomposable continuum and if R is the ray that is dense in the inverse limit, $\overline{R} - R$ is a proper subcontinuum that results from the inverse limit on $[f_s(s), s]$ using the restriction of f_s to that interval as the bonding map. We refer to $\varprojlim \mathbf{f_s}|[\mathbf{f_s}(\mathbf{s}), \mathbf{s}]$ as the *core* of $\varprojlim \mathbf{f_s}$ and the map $f_s|[f_s(s), s]$ as a *tent core*. Sometimes the tent core is rescaled to be the map of $[0, 1]$ onto itself given by $f_s(x) = sx + 2 - s$ for $0 \leq x \leq 1 - 1/s$ and $f_s(x) = s - sx$ for $1 - 1/s \leq x \leq 1$. Since the critical point is different depending on one's perspective, it is simply denoted by c in the remainder of this section.

Raines' approach to the case that the orbit of the critical point c is infinite has been to look at the omega limit set of c, $\omega(c) = \cap_{n=1}^{\infty} \overline{\{f^k(c) \mid k \geq n\}}$. When the orbit of c is infinite, $\omega(c) = [0, 1]$ or $\omega(c)$ is totally disconnected. If the orbit is infinite and $\omega(c)$ is totally disconnected, $\omega(c)$ may be a countable set, a Cantor set, or the union of a countable set and a Cantor set. It is in the case that $\omega(c) = [0, 1]$ that the Barge, Brucks and Diamond phenomenon of [2] occurs (i.e., there are parameter values at which the inverse limit of the tent map contains a copy of every continuum that arises as an inverse limit space from a tent family core).

Problem 5 (Raines) *Suppose f is a tent core with critical point c such that $\omega(c) = [0, 1]$. If C is a composant of $\varprojlim \mathbf{f}$, does C contain a copy of every continuum that arises as an inverse limit space of a tent family core?*

Problem 6 (Raines) *Suppose f is a unimodal map with critical point c. Give necessary and sufficient conditions on c so that $\varprojlim \mathbf{f}$ contains a copy of every continuum that arises as an inverse limit space in a tent family core.*

In case f is a tent core with critical point c and $\omega(c)$ is countable or the union of a countable set and a Cantor set, it is known that the inverse limit is an indecomposable arc continuum without end points (by an *arc continuum* we mean a continuum such that every proper subcontinuum is an arc). Good, Knight, and Raines have shown [10] that there are uncountably many members of the tent family cores with $\omega(c)$ countable that have topologically different inverse limits.

In case f is a tent core with critical point c and $\omega(c)$ is a Cantor set, the inverse limit is indecomposable but it may have end points. If it has end points the set of end points is uncountable [5]. The subcontinua of $\varprojlim \mathbf{f}$ can be quite complicated as demonstrated in [4]. This gives rise to the next problem.

Problem 7 (Raines) *Let f be a tent core with critical point c and $\omega(c)$ a Cantor set. Classify all possible subcontinua of $\varprojlim \mathbf{f}$.*

We close this section with one final problem. If n is a positive integer and σ is a permutation on the set $\{1, 2, \ldots, n\}$, define a map $f_\sigma : [0, 1] \to [0, 1]$ in the following way: (1) for $1 \leq i \leq n$ let $a_i = (i-1)/(n-1)$, (2) let $f_\sigma(a_i) = a_{\sigma(i)}$, and (3) extend f_σ linearly to all of $[0, 1]$. We call a map so constructed a *permutation map.* These maps are all Markov maps and many interesting continua result as the inverse limit space based on a permutation map. In [18] the author began a study of the inverse limits spaces that result from using a permutation map in an inverse limit. By brute force, all continua arising from permutations based on 3, 4, or 5 elements were determined.

Problem 8 *Classify all continua arising from permutation maps.*

5 The Property of Kelley

A continuum M with metric d is said to have the *Property of Kelley* provided if $\varepsilon > 0$ there is a positive number δ such that if p and q are points of M and $d(p, q) < \delta$ and H is a subcontinuum of M containing p then there is a subcontinuum K of M containing q such that $\mathcal{H}(H, K) < \varepsilon$ ($\mathcal{H}$ deonotes the Hausdorff distance on the hyperspace of subcontinua $C(M)$). This property that we now call the Property of Kelley was introduced by J. Kelley in his study of hyperspaces, but it is a nice continuum approximation property in its own right. The author considered the property in [14], [19], and [20]. While presenting the results that appeared in [19] and [20] in seminar, the author was asked the following question by W. J. Charatonik.

Problem 9 (Charatonik) *Is there a characterization of the Property of Kelley in terms of the inverse limit representation of the continuum?*

The author briefly tried to distill a sufficient condition from the proofs in the papers in [19] and [20] but never found a satisfying theorem. Nonetheless, it would be of interest to be able to determine the presence of the Property of Kelley based on some easily checked conditions on the bonding maps in an inverse limit representation of the continuum. Private communication with W. J. Charatonik indicates that he and a student have obtained some sufficient conditions on an inverse limit sequence to guarantee that the inverse limit have the Property of Kelley.

Permutation maps were defined in Section 4. In [18] it was shown that if f is a permutation map based on a permutation on 3, 4, or 5 elements, then $\varprojlim \mathbf{f}$ has the Property of Kelley. This leads us to ask the following question.

Problem 10 *Do all permutation maps produce continua with the Property of Kelley?*

6 Inverse Limits with upper semi-continuous bonding functions

W. S. Mahavier introduced inverse limits with upper semi-continuous bonding functions in [29] but as inverse limits on closed subsets of $[0,1] \times [0,1]$. In that article he showed that inverse limits on closed subsets of $[0,1] \times [0,1]$ are inverse limits on $[0,1]$ using upper semi-continuous closed set valued functions as bonding functions. In a subsequent paper [22], Mahavier and the author extended the definition to the setting of inverse limits on compact Hausdorff spaces using upper semi-continuous closed set valued bonding functions. If Y is a compact Hausdorff space, 2^Y denotes the collection of all closed subsets of Y. If X and Y are compact Hausdorff spaces, a function $f : X \to 2^Y$ is called *upper semi-continuous* at the point x of X provided if O is an open set in Y that contains $f(x)$ then there is an open set U in X that contains x and $f(t)$ is a subset of O for every t in U. If $X_1, X_2, X_3, \ldots$ is a sequence of compact Hausdorff spaces and $f_1, f_2, f_3, \ldots$ is a sequence of upper semi-continuous functions such that $f_i : X_{i+1} \to 2^{X_i}$ for each i, by the inverse limit of the inverse sequence $\{X_i, f_i\}$ is meant the subset of $\prod_{i>0} X_i$ that contains the point $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ if and only if $x_i \in f(x_{i+1})$. The reader will note that in case the functions are single valued, this definition reduces to the usual definition of an inverse limit. Beyond the collection of chainable continua that occur with single valued bonding functions, many interesting examples of continua result from inverse limits on $[0,1]$ with upper semi-continuous bonding functions that cannot occur with single valued functions. Among these are the Hilbert cube, the Cantor fan, a 2-cell with a sticker, and the Hurewicz continuum H that has the property that if M is a metric continuum there is a subcontinuum K of H and a mapping of K onto M. The example that produces a 2-cell with an attached arc leads to the following problem.

Problem 11 *Is there an upper semi-continuous function $f : [0,1] \to 2^{[0,1]}$ such that $\varprojlim \mathbf{f}$ is a 2-cell?*

Admittedly, this problem is rather more specific than most in this article, but perhaps it can serve as a starting point for an interesting investigation of these new and different inverse limits.

We end this section with a problem inspired by considerations from Section 7. Some models in economics are not well-defined either forward in time or backward in time [7], [37]. Some models consist of the union of two mappings that have no point in common. Perhaps an investigation of these new inverse limits using these models would be helpful to economists as well as a way to begin work on our next problem.

Problem 12 *Suppose* $f : [0,1] \to 2^{[0,1]}$ *is an upper semi-continuous function that is the union of two mappings of* $[0,1]$. *What can be said about* $\varprojlim \mathbf{f}$*?*

7 Applications of Inverse Limits in Economics

An exciting recent development in inverse limits is the development of models in economics in which the state of the model at time t is related to its state at time $t+1$ by some non-invertible mapping f. A solution to the model is an infinite sequence $x_1, x_2, x_3, \ldots$ such that $f(x_{t+1}) = x_t$ for $t = 1, 2, 3, \ldots$. So the set of solutions is the inverse limit on the state space using the map f as a bonding map. These models have arisen in cash-in-advance models [25] and overlapping generations models [30,31] studied by various economists. These models generally fall into a category of models described by economists as having "backward dynamics" or as models involving "backward maps". Economists are interested in the inverse limit because it contains as its points all future states predicted by the model. The author acknowledges private correspondence with Judy Kennedy and Brian Raines used in the development of this section and appreciates the contribution of problems by both of them. The problems that they contributed are labeled below with their names. Of course, any errors or misstatement of problems are solely the responsibility of the author.

If $f : X \to X$ and $g : X \to X$ are maps of a topological space X, we say that f and g are *topologically conjugate* provided there is a homeomorphism $h : X \to X$ such that $f \circ h = h \circ g$. If f and g are topologically conjugate, a homeomorphism h such that $f \circ h = h \circ g$ is called a conjugacy.

Problem 13 (Kennedy-Stockman) *Suppose* $f : [0,1] \to [0,1]$ *and* $g : [0,1] \to [0,1]$ *are topologically conjugate. How does one* construct *a homeomorphism* h *so that* $f \circ h = h \circ g$*?*

This problem is of deserves attention independent of the interest by economists. For economists the *existence* of a conjugacy is not sufficient information for carrying out some of the computations they need such as the computation of measures and then integrals for utility functions. Specific questions related to this problem and asked by Kennedy and Stockman include:

(1) Can the conjugacy be constructed by means of a sequence of approximations?
(2) If f and g are piecewise differentiable, must the conjugacy be piecewise differentiable?

The next problem is related to Problem 5 above.

Problem 14 (Kennedy-Stockman) *Do continua that contain copies of every inverse limit that arises in a tent family core occur as inverse limits in the cash-in-advance model [25] or the overlapping generations model [32]?*

Some models in economics are based on relations instead of functions so neither forward nor backward dynamics is well defined. In particular the Christiano-Harrison model [7] and a Stockman model [37] fit this scenario. Perhaps inverse limits with upper semi-continuous bonding functions (see Section 6) could be employed in an analysis of these models. Consequently, we reiterate **Problem 12**.

In considering models in economics, measure theory will likely play an important role for several reasons one of which we have already mentioned. For instance, when economists consider models involving backward dynamics, they would like to be able to "rank" the inverse limit spaces in some meaningful way, perhaps by using "natural" invariant measures. For a survey of literature on such measures see [12]. When comparing two inverse limit spaces but with a precise meaning of "better" to be determined, Kennedy and Stockman ask the following.

Problem 15 (Kennedy-Stockman) *Suppose policy A and policy B in economics lead to different inverse limit spaces. Determine which of the inverse limits is "better".*

With a precise meaning of "complex" to be determined, they also ask.

Problem 16 (Kennedy-Stockman) *For an economics model, what is the measure of the set of initial conditions that lead to "complex" dynamics?*

Problem 17 (Kennedy-Stockman) *In a model from economics, if an equilibrium point (i.e., point in the inverse limit) is chosen at random, what is the probability that it is "complex"?*

One search for appropriate measures on the inverse limit space centers on somehow making use of measures already developed. Kennedy and Stockman have recently succeeded in "lifting" given measures for interval maps to measures on the corresponding inverse limit spaces although they remark that such measures on the inverse limit space apparently are already known, see [24]. For an introduction to measures for interval maps see [1, Sections 6.4–6.6].

See also [12]. Kennedy and Stockman ask if there exist other useful measures one might consider, particularly in non-chaotic situations.

Recall that if $f : X \to X$ is a mapping of a metric space and x is a point of X, then the ω-limit set of x is $\omega(x) = \cap_{i>0}\overline{\{f^m(x) \mid m \geq i\}}$. If A is a closed subset of X and $f[A] = A$, we call A an *invariant* set. If A is a closed invariant subset of X, then the *basin of attraction of* A is $\{x \in X \mid \omega(x) \subset A\}$. A subset B of X is *nowhere dense* in X provided $\overline{B}$ does not contain an open set. A subset M of X is said to be *residual* in X provided $X - M$ is the union of countably many nowhere dense subsets. A closed invariant subset of X is called a *toplogical attractor* [32] for f provided the basin of attraction for A contains a residual subset of X and if A' is another closed invariant subset of X then the common part of the basin of attraction of A' and the basin of attraction of A is the union of at most countably many nowhere dense sets. For more information of topological attractors and metric attractors (defined below), see [32].

One possible tool for analyzing an inverse limit arising in a model from economics lies in the shift homeomorphism. There are two shifts and they are inverses of each other. Specifically, below we are referring to the shift $\sigma : \varprojlim \mathbf{f} \to \varprojlim \mathbf{f}$ given by $\sigma(\mathbf{x}) = (x_2, x_3, x_4, \ldots)$. Raines asks the following.

Problem 18 (Raines) *Let f be a map of the interval. Find necessary and sufficient conditions for $\varprojlim \mathbf{f}$ to admit a proper subset that is a topological attractor for the shift homeomorphism.*

Problem 19 (Raines) *Let f be a unimodal map of the interval. Classify all of the topological attractors for the shift homeomorphism on $\varprojlim \mathbf{f}$.*

Not all models from economics involve one-dimensional spaces. This prompts the following problem.

Problem 20 (Raines) *Let f be a map of $[0,1] \times [0,1]$. Identify topological attractors in $\varprojlim \mathbf{f}$ under the shift homeormorphism.*

If X is a metric space with a measure μ, $f : X \to X$ is a mapping and A is a closed invariant subset of X, then A is called a *metric attractor* for f provided the basin of attraction for A has positive measure and and if A' is another closed invariant subset of X then the common part of the basin of attraction of A' and the basin of attraction of A has measure zero.

Problem 21 (Raines) *In the previous two problems, change the phrase topological attractor to metric attractor.*

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