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INVERSE LIMITS WITH SET-VALUED FUNCTIONS HAVING GRAPHS THAT ARE SINUSOIDS

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ABSTRACT. Siinusoids are subcontinua of the unit square similar to the standard $\sin(1/x)$-curve. We show that inverse limits with a sequence of set-valued functions having graphs that are sinusoids are chainable continua. In the process we prove that if M and N are continua and $f : M \twoheadrightarrow N$ is a monotone mapping such that point inverses under f are C-sets with property P and the continuum N has property P then M has property P where P is any one of the properties: (1) atriodic, (2) hereditarily decomposable, (3) hereditarily unicoherent, and (4) hereditarily decomposable chainable.

1. INTRODUCTION

In 1955 A. D. Wallace introduced the notion of a C-set and explored C-sets in the context of topological semigroups. In 1982 the author [4] investigated C-sets and their role in continuum theory. In the present article, we make use of C-sets to obtain results about inverse limits with upper semi-continuous bonding functions. Specifically, here we show in Theorem 4.3 that inverse limits on $[0, 1]$ with upper semi-continuous set-valued bonding functions having graphs that are sinusoids are chainable continua. In an earlier paper, [6, Example 5.4], we showed that the inverse limit on $[0, 1]$ with an upper semi-continuous bonding function whose graph is piecewise linear version of a "standard" $\sin(1/x)$-curve is a chainable continuum. Soon thereafter James P. Kelly, [11], extended that result

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to inverse limits on $[0,1]$ with a single upper semi-continuous function having a graph that is what he calls an irreducible function. Here our approach returns to some of the original ideas of [6]. Initially, we tried mimicking that argument but the results obtained were not as general as the techniques employed here actually produce. We rely on Bing's characterization of chainability in hereditarily decomposable continua as continua that are atriodic and hereditarily unicoherent, and it allows us to permit sequences of bonding functions that do not have to be constant. This required an investigation found in Section 3 of monotone mappings having point inverses that are C-sets. Over the years, there has been considerable interest in monotone maps on continua where the image is a chainable continuum. Notably, such a question appeared in the Houston Problem Book couched in the language of upper semi-continuous decompositions, see [2, Problem 105]. For monotone mappings it amounts to asking if an atriodic 1-dimensional continuum is chainable if its image under a monotone mapping with chainable point-inverses is chainable. This question was answered in the negative in a paper by James F. Davis and the author, [3]. The example in that paper is a nonchainable indecomposable continuum with a monotone mapping to a chainable continuum having only one nondegenerate point-inverse and it is an arc. Interestingly, that point-inverse is not a C-set in the nonchainable continuum. In this light, perhaps the following question is of interest.

Question 1.1. *If M is a 1-dimensional atriodic continuum and f is a monotone mapping of M onto a chainable continuum N such that point-inverses are chainable C-sets in M, is M chainable?*

Theorem 3.8 yields that the answer is "yes" in case N as well as all point inverses are hereditarily decomposable chainable continua even without assuming M is atriodic and 1-dimensional.

2. Definitions and Notation

A *compactum* is a compact metric space; a *continuum* is a connected compactum. If X is a compactum, 2^X denotes the collection of all compact subsets of X. If each of X and Y is a compactum, a function $f : X \to 2^Y$, herein denoted $f : X \nearrow Y$, is said to be *upper semi-continuous at the point x of X* provided that if V is an open subset of Y that contains $f(x)$ then there is an open subset U of X containing x such that if t is a point of U then $f(t) \subseteq V$. A function $f : X \nearrow Y$ is called *upper semi-continuous* provided it is upper semi-continuous at each point of X. If $f : X \nearrow Y$ is a set-valued function, by the *graph* of f, denoted $G(f)$, we mean $\{(x,y) \in X \times Y \mid y \in f(x)\}$; if $f : X \nearrow Y$ and $g : Y \nearrow Z$, then $g \circ f : X \nearrow Z$ denotes the function given by $z \in g \circ f(x)$ if and only if there is a point y of Y such that $y \in f(x)$ and $z \in g(y)$. It is known that if X and Y are compacta and M is a subset of $X \times Y$ such that X is the projection of M to its set of first coordinates then M is closed if and only if M is the graph of an upper semi-continuous function [10, Theorem 2.1] or [5, Theorem 1.2, p. 3]. If $\boldsymbol{s} = s_1, s_2, s_3, \ldots$ is a sequence, we normally denote the sequence in boldface type and its terms in italics. Suppose $\boldsymbol{X}$ is a sequence of compacta and $f_n : X_{n+1} \nearrow X_n$ is an upper semi-continuous function for each $n \in \mathbb{N}$. By the *inverse limit* of $\boldsymbol{f}$, denoted $\varprojlim \boldsymbol{f}$, we mean $\{\boldsymbol{x} \in \prod_{i>0} X_i \mid x_i \in f_i(x_{i+1})$ for each positive integer $i\}$. If $\{X_a \mid a \in D\}$ is a collection of sets and A is a subset of D, we denote by π_A the natural projection of $\prod_{a \in D} X_a$ onto $\prod_{a \in A} X_a$. If a and b are two numbers, we denote the interval with endpoints a and b by $[a, b]$ whether or not a is smaller. A continuum M is *hereditarily unicoherent* provided if A and B are subcontinua of M with a point in common then $A \cap B$ is connected. A continuum M is a *triod* provided there is a subcontinuum H of M such that $M - H$ has three components; a continuum is *atriodic* provided it does not contain a triod. A continuum homeomorphic to an inverse limit on intervals with mappings is called a *chainable* continuum.

A subset K of a continuum M is a *C-set* in M provided it is true that if H is a subcontinuum of M containing a point of K and a point of $M - K$ then $K \subseteq H$. A subcontinuum C of a continuum M is said to be *terminal* in M provided if H and K are subcontinua of M each intersecting C then $H \subseteq K \cup C$ or $K \subseteq H \cup C$.

The following theorems with proofs or citations to original sources may be found in [6].

Theorem 2.1. *If A and B are chainable continua and $A \cap B$ is a continuum that is terminal and a C-set in both A and B, then $A \cup B$ is chainable. Moreover, $A \cap B$ is a C-set in $A \cup B$.*

Theorem 2.2. *Suppose H is a subcontinuum of the continuum M and K is a C-set in H. If there is an open subset U of M such that $K \subseteq U \subseteq H$, then K is a C-set in M.*

In [1], R H Bing proved the following theorem characterizing chainability among hereditarily decomposable continua.

Theorem 2.3. (Bing) *An hereditarily decomposable continuum is chainable if and only if it is atriodic and hereditarily unicoherent.*

A set traditionally used in the proof that $\varprojlim \boldsymbol{f}$ is nonempty and compact is $\{\boldsymbol{x} \in \prod_{k>0} X_k \mid x_i \in f_i(x_{i+1})$ for $1 \leq i \leq n\}$. Because this set was originally denoted G_n, we adopt and use throughout this article the notation $G'_n = \{\boldsymbol{x} \in \prod_{k=1}^{n+1} X_k \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$ for the projection of G_n into the product of the first $n+1$ factor spaces, i.e., $G'_n = \pi_{\{1,2,\ldots,n+1\}}(G_n)$. These sets G'_n are precisely the "approximations" whose connectedness characterizes the connectedness of the inverse limit. In the literature G'_n has been denoted by $G'(f_1, f_2, \ldots, f_n)$.

Finally, in Section 4 we make use of the following theorem from [6, Corollary 4.3]. Theorem 2.4 also follows from [7, Corollary 4.2] and the fact that an inverse limit on chainable continua with bonding functions that are mappings is chainable.

Theorem 2.4. *Suppose $\boldsymbol{X}$ is a sequence of continua and $f_n : X_{n+1} \nearrow X_n$ is an upper semi-continuous function for each positive integer n. If G'_n is a chainable continuum for each positive integer n then $\varprojlim \boldsymbol{f}$ is a chainable continuum.*

3. C-sets and monotone maps

In this section we develop some results about monotone mappings that have point inverses that are C-sets and use them to prove Theorem 3.8. If these results are in the literature the author is not aware of it. However, they are simple to prove and we include them for completeness. We make use of these lemmas and theorems in the next section of this paper.

Lemma 3.1. *Suppose each of M and N is a continuum and $f : M \twoheadrightarrow N$ is a mapping. If x is a point of N such that $f^{-1}(x)$ is a C-set in M and H is a subcontinuum of M containing a point of $f^{-1}(x)$ such that $f(H)$ is nondegenerate then $f^{-1}(x)$ is a subset of H.*

Proof. Because $f(H)$ is nondegenerate there is a point of H not in $f^{-1}(x)$. Thus H is a subcontinuum of M contining a point of $f^{-1}(x)$ and a point not in $f^{-1}(x)$. Because $f^{-1}(x)$ is a C-set in M, H contains it. □

Lemma 3.2. *Suppose each of M and N is a continuum and $f : M \twoheadrightarrow N$ is a monotone mapping such that $f^{-1}(x)$ is a C-set in M for each x in N. If H is a subcontinuum of M and $f(H)$ is nondegenerate, then $f^{-1}(f(H)) = H$.*

Proof. We only need to show that each point of $f^{-1}(f(H))$ is in H. If $x \in f^{-1}(f(H))$ then $f(x) \in f(H)$ so there is a point y of H such that $f(y) = f(x)$. Then, y is a point of $f^{-1}(x)$. Because H contains a point of $f^{-1}f((x))$ and $f(H)$ is nondegenerate, by Lemma 3.1 H contains $f^{-1}(f(x))$ so $x \in H$. □

Our next theorem will be useful in the next section of this article.

Theorem 3.3. *Suppose a and b are numbers with $a < b$, M is a continuum and $f : M \twoheadrightarrow [a, b]$ is a monotone mapping such that $f^{-1}(t)$ is a C-set in M for each t in $[a, b]$. Then, $f^{-1}(a)$ and $f^{-1}(b)$ are terminal in M.*

Proof. Because f is monotone, $f^{-1}(a)$ is a continuum. Suppose H and K are subcontinua of M each intersecting $f^{-1}(a)$. We may assume that neither is a subset of $f^{-1}(a)$. By Lemma 3.1 $f^{-1}(a)$ is a subset of $H \cap K$. There exist c and d such that $f(H) = [a, c]$ and $f(K) = [a, d]$. Then, $f^{-1}([a, c]) \subseteq f^{-1}([a, d])$. But $H = f^{-1}([a, c])$ and $K = f^{-1}([a, d])$, so $H \subseteq K$. Then, $H \cup f^{-1}(a) \subseteq K \cup f^{-1}(a)$ so $f^{-1}(a)$ is terminal in M. Similarly, $f^{-1}(b)$ is terminal in M. □

Theorem 3.4. *Suppose M and N are continua, N is hereditarily decomposable, and $f : M \to N$ is a monotone mapping such that $f^{-1}(x)$ is a C-set in M for each x in N. If H is a subcontinuum of M such that $f(H)$ is nondegenerate, then H is decomposable.*

Proof. Suppose $f(H) = A \cup B$ where A and B are proper subcontinua of $f(H)$. Because f is monotone, $f^{-1}(A)$ and $f^{-1}(B)$ are subcontinua of M. By Lemma 3.2, $H = f^{-1}(f(H))$. Because $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, we see that $H = f^{-1}(A) \cup f^{-1}(B)$. Because A contains a point not in B , $f^{-1}(A)$ contains a point not in $f^{-1}(B)$. Because B contains a point not in A, $f^{-1}(B)$ contains a point not in $f^{-1}(A)$. Thus, H is the union of two proper subcontinua. □

Theorem 3.5. *Suppose M and N are continua, N is hereditarily decomposable, and $f : M \to N$ is a monotone mapping such that $f^{-1}(x)$ is an hereditarily decomposable C-set in M for each x in N. Then, M is hereditarily decomposable.*

Proof. Suppose H is a subcontinuum of M. If $f(H)$ is nondegenerate, by Theorem 3.4, H is decomposable. If $f(H) = \{x\}$, then H is a subset of $f^{-1}(x)$ which is hereditarily decomposable by hypothesis, so H is decomposable. □

Theorem 3.6. *Suppose M and N are continua, N is atriodic, and $f : M \to N$ is a monotone mapping such that $f^{-1}(x)$ is an atriodic C-set in M for each x in N. Then, M is atriodic.*

Proof. Suppose M contains a triod T. Let A be a subcontinuum of T such that $T - A$ has three components, T_1, T_2, and T_3. Let $H = T_1 \cup A$, $K = T_2 \cup A$, $L = T_3 \cup A$, and let p be a point of A. Then, H, K, and L are three subcontinua of M all containing the point p. If $f(H)$, $f(K)$, and $f(L)$ are degenerate then H, K, and L are subcontinua of the atriodic continuum $f^{-1}(f(p))$ contradicting the fact that T is a triod. Suppose

$f(H)$ is degenerate and one of $f(K)$ and $f(L)$, say $f(K)$, is nondegenerate. Then, by Lemma 3.1, $f^{-1}(f(p))$ is a subset of K. But, H is a subset of $f^{-1}(f(p))$, so H is a subset of K, again contradicting the fact that T is a triod.

If $f(H)$, $f(K)$, and $f(L)$ are all nondegenerate, then, because all three contain $f(p)$ and N is atriodic, one of them is a subset of the union of the other two. Suppose $f(H) \subseteq f(K) \cup f(L)$. By Lemma 3.2, $H = f^{-1}(f(H))$, $K = f^{-1}(f(K))$, and $L = f^{-1}(f(L))$. But, $f^{-1}(f(H)) \subseteq f^{-1}(f(K) \cup f(L)) = f^{-1}(f(K)) \cup f^{-1}(f(L)) = K \cup L$. This is a contradiction so M does not contain a triod. □

Theorem 3.7. *Suppose M and N are continua, N is hereditarily unicoherent, and $f : M \to N$ is a monotone mapping such that $f^{-1}(x)$ is an hereditarily unicoherent C-set in M for each x in N. Then, M is hereditarily unicoherent.*

Proof. Suppose H and K are subcontinua of M with a common point, p. If $f(H)$ and $f(K)$ are degenerate, then H and K are subcontinua of the hereditarily unicoherent subcontinuum $f^{-1}(f(p))$ of M, so $H \cap K$ is connected. If $f(H)$ is nondegenerate, by Lemma 3.1, $f^{-1}(f(p))$ is a subset of H. If $f(K)$ is degenerate, $K \subseteq f^{-1}(f(p))$, so $H \cap K = K$. If both $f(H)$ and $f(K)$ are nondegenerate, then, because N is hereditarily unicoherent, $f(H) \cap f(K)$ is connected. Because f is monotone, $f^{-1}(f(H) \cap f(K))$ is connected. But, $f^{-1}(f(H) \cap f(K)) = f^{-1}(f(H)) \cap f^{-1}(f(K)) = H \cap K$. □

Using Bing's theorem (Theorem 2.3) and Theorems 3.5, 3.6, and 3.7 from this section, we have the following theorem.

Theorem 3.8. *Suppose M is a continuum, N is an hereditarily decomposable chainable continuum, and $f : M \twoheadrightarrow N$ is a monotone mapping such that $f^{-1}(x)$ is an hereditarily decomposable chainable C-set in M for each x in N. Then, M is chainable.*

4. Sinusoids

In this section we define a $\sin(1/x)$-type structure we call a sinusooid. Sinusoids include a traditional $\sin(1/x)$-curve and the curve shown by Dorothy Sherling [12] not to be homeomorphic to an inverse limit on intervals with a single mapping. Here we show that inverse limits with a sequence of sinusoids is a chainable continuum generalizing the result in [6, Example 5.4]. This investigation was prompted by a question asked by Rob Roe in private correspondence with the author. Specifically, in the framework established below, Roe asked if the sinusoid determined by the sequence $1, 0, 1/2, 0, 1, 0, 1/2, 0, 1, \ldots$ produces a chainable continuum in the inverse limit. The result herein answers this question in the affirmative. We begin with a lemma.

Lemma 4.1. *Suppose* $\boldsymbol{X}$ *is a sequence of continua and* $\boldsymbol{f}$ *is a sequence of upper semi-continuous functions such that* $f_i : X_{i+1} \to C(X_i)$ *for each positive integer* i*. Then, for each positive integer* n*,* $\pi_{n+1}|G'_n$ *is monotone.*

Proof. We proceed by induction. Let x be a point of X_2. Then, $\{(y, x) \in G'_1 \mid y \in f_1(x)\} = f_1(x) \times \{x\}$. Because $f_1(x)$ is connected, $f_1(x) \times \{x\} = \pi_2^{-1}(x) \cap G'_1 = (\pi_2|G'_1)^{-1}(x)$ is connected.

Suppose k is a positive integer such that $\pi_{k+1}|G'_k$ is monotone and let x be a point of X_{k+2}. Then, $f_{k+1}(x)$ is a subcontinuum of X_{k+1}; consequently, $(\pi_{k+1}|G'_{k+1})^{-1}(f_{k+1}(x))$ is a continuum. However, this set is homeomorphic to $\{(x_1, x_2, \ldots, x_{k+2}) \in G'_{k+1} \mid x_{k+2} = x\}$, but $\{(x_1, x_2, \ldots, x_{k+2}) \in G'_{k+1} \mid x_{k+2} = x\} = (\pi_{k+2}|G'_{k+1})^{-1}(x)$. Thus, $\pi_{k+2}|G'_{k+1}$ is monotone. □

Suppose $z_0, z_1, z_2, \ldots$ is a sequence of numbers from $[0, 1]$ such that

(1) $z_0 = 1$
(2) $z_{i+1} > z_i$ if i is odd and $z_{i+1} < z_i$ otherwise
(3) some subsequence of $\boldsymbol{z}$ converges to 0 and another subsequence of $\boldsymbol{z}$ converges to 1.

Let $f : [0, 1] \nearrow [0, 1]$ be the upper semi-continuous function defined as follows:

(1) $f(0) = [0, 1]$
(2) $f(1/2^i) = z_i$ for $i = 0, 1, 2, \ldots$
(3) f is a homeomorphism on $[1/2^i, 1/2^{i-1}]$ for each i.

We call $G(f)$ the *sinusoid determined by* $\boldsymbol{z}$, or, simply, a *sinusoid.* The condition that $\boldsymbol{z}$ contains subsequences converging to 0 and 1 ensure that $\{0\} \times [0, 1]$ is a C-set in $G(f)$. The condition that $z_0 = 1$ is artificial, but we assume it for convenience. It is only necessary that z_0 be positive.

Theorem 4.2. *Suppose* $\boldsymbol{f}$ *is a sequence of upper semi-continuous functions such that, for each positive integer* $i, f_i : [0, 1] \nearrow [0, 1]$ *has a graph that is a sinusoid. Then, for each positive integer* n, G'_n *is an hereditarily decomposable chainable continuum.*

Proof. Observe that, for each positive integer i, $f_i : [0, 1] \to C([0, 1])$ so, by Lemma 4.1, $\pi_{n+1}|G'_n$ is monotone for each positive integer n.

We proceed by induction. Note that if $0 \le a < b \le 1$, then $\{(x_1, x_2) \in G'_1 \mid a \le x_2 \le b\}$ is an hereditarily decomposable chainable continuum such that

(1) if $a \le t \le b$ then $\{(x_1,x_2) \in G'_1 \mid x_2 = t\}$ is a point or an interval that is a C-set in $\{(x_1,x_2) \in G'_1 \mid a \le x_2 \le b\}$ and
(2) $\{(x_1,x_2) \in G'_1 \mid x_2 = a\}$ and $\{(x_1,x_2) \in G'_1 \mid x_2 = b\}$ are terminal in $\{(x_1,x_2) \in G'_1 \mid a \le x_2 \le b\}$.

Suppose k is a positive integer such that if $0 \le a < b \le 1$ then $\{\boldsymbol{x} \in G'_k \mid a \le x \le b\}$ is an hereditarily decomposable chainable continuum such that

(1) if $a \le t \le b$ then $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = t\}$ is a continuum that is a C-set in $\{\boldsymbol{x} \in G'_k \mid a \le x_{k+1} \le b\}$ and
(2) $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = a\}$ and $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = b\}$ are terminal in $\{\boldsymbol{x} \in G'_k \mid a \le x_{k+1} \le b\}$.

Denote f_{k+1} by f and suppose $\boldsymbol{z}$ is the sequence such that $G(f)$ is the sinusoid determined by $\boldsymbol{z}$. Let $\varphi_0 : [0,1] \to [0,1]$ be the mapping given by $\varphi_0(t) = 0$ for each t. For each positive integer i, let $\varphi_i : [z_{i-1}, z_i] \twoheadrightarrow [1/2^i, 1/2^{i-1}]$ be the surjective homeomorphism that is the inverse of the restriction of f to $[1/2^i, 1/2^{i-1}]$. Note that $\varphi_i(z_i) = 1/2^i$ for each $i \ge 0$ and $G(f^{-1}) = \varphi_0 \cup \varphi_1 \cup \varphi_2 \cup \cdots$. For $i \ge 0$, define $\Phi_i : \{\boldsymbol{x} \in G'_k \mid x_{k+1} \in [z_{i-1}, z_i]\} \to G'_{k+1}$ by $\Phi_i(\boldsymbol{x}) = (x_1, x_2, \ldots, x_{k+1}, \varphi_i(x_{k+1}))$. For $i \ge 0$ let $L_i = \Phi_i(\{\boldsymbol{x} \in G'_k \mid x_{k+1} \in [z_{i-1}, z_i]\})$. Because φ_i is a mapping for each $i \ge 0$, Φ_i is a homeomorphism so each L_i is an hereditarily decomposable chainable continuum. Furthermore, $L_i \cap L_{i+1} = \{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = \varphi_i(z_i)\}$, $L_i \cap L_j \ne \emptyset$ if and only if $j \in \mathbb{N}$ and $|i - j| \le 1$, and $G'_{k+1} = L_0 \cup L_1 \cup L_2 \cup \cdots$. As a consequence of the fact that the sequence $\boldsymbol{z}$ contains subsequences converging to both 0 and 1, $L_0 = \mathrm{cl}(L_1 \cup L_2 \cup L_3 \cup \cdots) - (L_1 \cup L_2 \cup L_3 \cup \cdots)$.

Because $\pi_{k+2}|G'_{k+1}$ is a monotone map from G'_{k+1} onto $[0,1]$, if $0 \le a < b \le 1$, then $\{\boldsymbol{x} \in G'_{k+1} \mid a \le x_{k+2} \le b\}$ is a continuum.

We now show that if $0 \le a \le t \le b \le 1$ then $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = t\}$ is a C-set in $\{\boldsymbol{x} \in G'_{k+1} \mid a \le x_{k+2} \le b\}$. We consider cases:

(1) $t = 1$,
(2) $t \in [1/2^j, 1/2^{j-1}]$ for some positive integer j, and
(3) $t = 0$.

Because $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = 1\}$ is degenerate, it is a C-set in any continuum containing it. If $1/2^j < t < 1/2^{j-1}$ there exist c, d with $1/2^j < c < t < d < 1/2^{j-1}$ such that the segment (c,d) is a subset of the segment (a,b). By the inductive hypothesis, $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = \varphi_j^{-1}(t)\}$ is a C-set in $\{\boldsymbol{x} \in G'_k \mid \varphi_j^{-1}(c) \le x_{k+1} \le \varphi_j^{-1}(d)\}$. Because Φ_j is a homeomorphism, $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = t\}$ is a C-set in $\{\boldsymbol{x} \in G'_{k+1} \mid c \le x \le d\}$; because it lies in the open set $\{\boldsymbol{x} \in G'_{k+1} \mid c < x_{k+2} < d\}$ lying in the subcontinuum $\{\boldsymbol{x} \in G'_{k+1} \mid c \le x \le d\}$ of $\{\boldsymbol{x} \in G'_{k+1} \mid a \le x \le b\}$, by Theorem 2.2 it is a C-set in $\{\boldsymbol{x} \in G'_{k+1} \mid a \le x \le b\}$. In case t is one of the endpoints of $[1/2^j, 1/2^{j-1}]$, say $t = 1/2^j$, choose c and d so that $1/2^{j+1} < c < 1/2^j < d < 1/2^{j-1}$ while keeping the segment (c,d) a subset of the segment (a,b). Because $\{\boldsymbol{x} \in G'_k \mid f(c) \le x_{k+1} \le z_j\}$ is a chainable continuum having $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = z_j\}$ as a terminal subcontinuum and Φ_{j-1} is a homeomorphism, it follows that $\{\boldsymbol{x} \in G'_{k+1} \mid c \le x_{k+2} \le 1/2^j\}$ is a chainable continuum having $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = 1/2^j\}$ as a terminal subcontinuum. Similarly, $\{\boldsymbol{x} \in G'_{k+1} \mid 1/2^j \le x_{k+2} \le d\}$ is a chainable continuum having $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = 1/2^j\}$ as a terminal subcontinuum. Thus, $\{\boldsymbol{x} \in G'_{k+1} \mid c \le x_{k+2} \le 1/2^j\}$ and $\{\boldsymbol{x} \in G'_{k+1} \mid 1/2^j \le x_{k+2} \le d\}$ are chainable continua intersecting in a common terminal continuum, $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = 1/2^j\}$. By Theorem 2.1, $\{\boldsymbol{x} \in G'_{k+1} \mid x_{k+2} = 1/2^j\}$ is a C-set in $\{\boldsymbol{x} \in G'_{k+1} \mid c \le x_{k+2} \le d\}$. Because this C-set lies in the open set $\{\boldsymbol{x} \in G'_{k+1} \mid c \le x_{k+2} \le 1/2^j\}$ lying in the continuum $\{\boldsymbol{x} \in G'_{k+1} \mid c \le x_{k+2} \le d\}$, it is a C-set in $\{\boldsymbol{x} \in G'_{k+1} \mid a \le x_{k+2} \le b\}$. Finally, in case $t = 0$, let H be a subcontinuum of $\{\boldsymbol{x} \in G'_{k+1} \mid 0 \le x_{k+2} \le b\}$ that contains a point of L_0 and a point not in L_0. Then, H contains a point $\boldsymbol{p}$ such that $p_{k+2} > 0$. There is a positive integer N such that $1/2^i < p_{k+2}$ for $i \ge N$. Then, for each $i \ge N$, H contains $\pi^{-1}_{k+2}(t)$ for $t \in [1/2^{i+1}, 1/2^i]$, a C-set in $\{\boldsymbol{x} \in G'_{k+1} \mid a \le x_{k+2} \le b\}$. So $L_i \subseteq H$ for $i \ge N+1$. Therefore, H contains $L_0 = \pi^{-1}_{k+2}(0)$ so we have $\pi^{-1}_{k+2}(0)$ is a C-set in $\{\boldsymbol{x} \in G'_{k+1} \mid a \le x_{k+2} \le b\}$.

That $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = a\}$ and $\{\boldsymbol{x} \in G'_k \mid x_{k+1} = b\}$ are terminal in $\{\boldsymbol{x} \in G'_k \mid a \le x_{k+1} \le b\}$ is a consequence of Theorem 3.3.

Thus we have that $\pi_{k+2}|G'_{k+1}$ is a monotone map of G_{k+1} onto $[0,1]$ such that $(\pi^{-1}_{k+2}|G'_{k+1})^{-1}(t)$ is an hereditarily decomposable chainable C-set in G'_{k+1} for each t in $[0,1]$. By Theorem 3.8, G'_{k+1} is chainable and the induction is complete. □

Using Theorem 4.2 along with Theorem 2.4 we have the following theorem.

Theorem 4.3. *Suppose $\boldsymbol{f}$ is a sequence of upper semi-continuous functions such that, for each positive integer i, $f_i : [0,1] \nearrow [0,1]$ has a graph that is a sinusoid. Then $\varprojlim \boldsymbol{f}$ is chainable.*

There are several natural conditions to impose on upper semi-continuous set-valued functions on $[0,1]$ that could be considered generalizations of

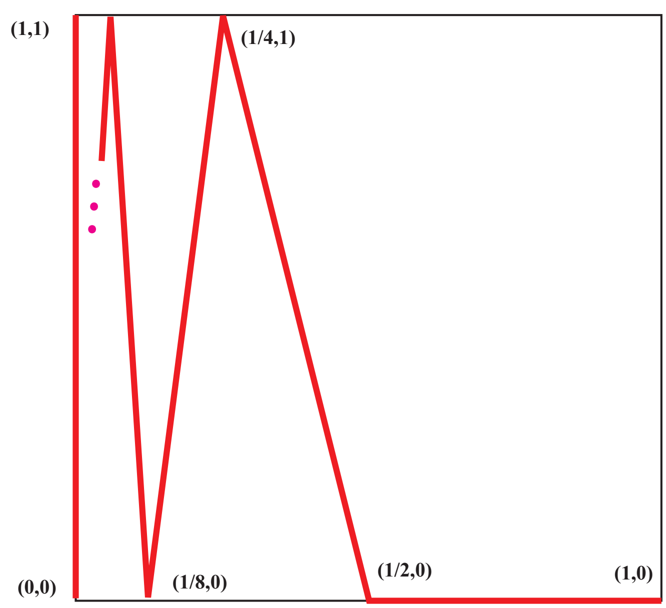

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Figure 1. The graph of the bonding function in Example 4.5

mappings. These include those whose graphs are arcs as well as interval-valued functions. One condition that may have been overlooked is requiring the function to have values that are C-sets in its graph. With that in mind we pose the following problem.

Problem 4.4. *Suppose* $\boldsymbol{f}$ *is a sequence of upper semi-continuous functions such that, for each positive integer* i, $f_i : [0,1] \to C([0,1])$ *and* $f(t)$ *is a* C*-set in* $G(f)$ *for each* $t \in [0,1]$*. If* $\varprojlim \boldsymbol{f}$ *is treelike, is* $\varprojlim \boldsymbol{f}$ *chainable?*

In Problem 4.4, some condition restricting the behavior of "flat spots" on the graphs is needed. We conclude with an example that illustrates this. We imposed the condition that $\varprojlim \boldsymbol{f}$ be treelike because treelikeness of is well understood due to Marsh's characterization as found in [14].

Example 4.5. *Let f be the upper semi-continuous set-valued function on $[0,1]$ obtained by replacing the homeomorphism on $[1/2,1]$ in the sinusoid determined by the sequence $1,0,1,0,\ldots$ with the map that has value 0 for each point of $[1/2,1]$. Then $\varprojlim \boldsymbol{f}$ contains a copy of the Hilbert cube, $[1/2,1]\times\{0\}\times[1/2,1]\times\{0\}\times\cdots$, and so is infinite dimensional. See Figure 1 for $G(f)$.*

A closer examination of Example 4.5 shows that, even though $f(t)$ is a C-set in $G(f)$ for each $t\in[0,1]$ and $\pi_3|G_2'$ is monotone, $(\pi_3|G_2')^{-1}(t)$ is not a C-set in G_2' for any $t\in[1/2,1]$ precisely because of the "flat spot" on $G(f)$. Difficulties regarding dimension and treelikeness of inverse limits on $[0,1]$ with set-valued functions that are caused by "flat spots" on the graphs of the bonding functions are discussed in articles like [8], [13], and [14].

References

[1] R H Bing, *Snake-like continua*, Duke J. Math. **18**(1951), 653–663.

[2] Howard Cook, W. T. Ingram, and Andrew Lelek, *A list of problems known as the Houston Problem Book*, Continua, Lecture Notes in Pure and Appl. Math., vol. 170, Dekker, New York, 1995, 365–398.

[3] James F. Davis and W. T. Ingram, *An atriodic tree-like continuum with positive span which admits a monotone mapping to a chainable continuum*, Fund. Math. **131**(1988), 13–24.

[4] W. T. Ingram, *C-sets and mappings of continua*, Topology Proc. **7**(1982), 83–90.

[5] W. T. Ingram, An Introduction to Inverse Limits with Set-valued Functions, Springer Briefs, Springer, New York, 2012. http://dx.doi.org/10.1007/978-1-4614-4487-9

[6] W. T. Ingram, *Concerning chainability of inverse limits on* $[0,1]$ *with set-valued bonding functions*, Topology Proc. **42** (2013), 327–340.

[7] W. T. Ingram, *Inverse limits with families of set-valued functions*, Bol. Soc. Mat. Mex., **21**(2015), 53–70, dx.doi.org/10.1007/s40590-014-0017-7, (2014).

[8] W. T. Ingram, *Concerning dimension and tree-likeness of inverse limits with set-valued functions*, Houston J. Math. **40**(2014), 621–632.

[9] W. T. Ingram, *One-dimensional inverse limits with set-valued functions*, Topology Proc. **46** (2015), 243–253.

[10] W. T. Ingram and William S. Mahavier, Inverse Limits: From Continua to Chaos, Advances in Mathematics vol. 25, Springer, New York, 2012.

[11] James P. Kelly, *Chainability of inverse limits with a single irreducible function on* $[0,1]$, Topology Appl. **176**(2014), 57–75.

[12] Dorothy S. Marsh, *A chainable continuum not homeomorphic to an inverse limit on* $[0,1]$ *with only one bonding map*, Colloq. Math. **43**(1980), 75–80 **MR**615973.

[13] M. M. Marsh, *Tree-like inverse limits on* $[0,1]$ *with interval-valued functions*, Topology Proc. **48**(2016), 215–232.

[14] M. M. Marsh, *A characterization of tree-like inverse limits on* $[0,1]$ *with interval-valued functions*, **50**(2017), 101–109.

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