2-2 Separable Variables Examples.

Solve the given DEs by separation of variables.

1. $\frac{dy}{dx} + 2xy^2 = 0$

$\frac{dy}{dx} = -2xy^2$

$\int \frac{dy}{y^2} = \int -2x\,dx$

$-y^{-1} = -x^2 + c$

$y(x) = \frac{1}{x^2+c}$

2. $e^x y \frac{dy}{dx} = e^{-y}(1+e^{-2x})$

$\int e^y y\,dy = \int (e^{-x} + e^{-3x})\,dx$

$u = y \qquad dv = e^y$

$du = dy \qquad v = e^y$

$ye^y - \int e^y dy = -e^{-x} - \frac{1}{3}e^{-3x} + c$

$ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C$

$ye^y - e^y + e^{-x} + \frac{1}{3}e^{-3x} = C.$

3. $\csc y\,dx + \sec^2 x\,dy = 0$

$\sec^2 x\,dy = -\csc x\,dy$

$\int \frac{dy}{\csc y} = -\int \frac{dx}{\sec^2 x}$

$\int \sin y\,dy = -\int \cos^2 x\,dx$

$+\cos y = +\int \frac{1+\cos 2x}{2}\,dx$

$\cos y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

$4\cos y = 2x + \sin 2x + C_1$

4. $\dfrac{dy}{dx} = \dfrac{xy+2y-x-2}{xy-3y+x-3} = \dfrac{(y-1)(x+2)}{(y+1)(x-3)}$

$$\int\left(\frac{y+1}{y-1}\right)dy = \int\left(\frac{x+2}{x-3}\right)dx.$$

$$\begin{array}{r|l} & 1+\frac{2}{y-1} \\ \hline y-1 & y+1 \\ & \underline{y-1} \\ & 2 \end{array} \qquad\qquad \begin{array}{r|l} & 1+\frac{5}{x-3} \\ \hline x-3 & x+2 \\ & \underline{x-3} \\ & 5 \end{array}$$

So $$\int\left(1+\frac{2}{y-1}\right)dy = \int\left(1+\frac{5}{x-3}\right)dx$$

$$y+2\ln|y-1| = x+5\ln|x-3|+C$$

Find an implicit solution and an explicit solution of the given IVP.

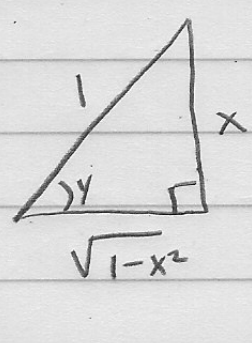
5. $\sqrt{1-y^2}\,dx = \sqrt{1-x^2}\,dx$; $y(0) = \frac{\sqrt{3}}{2}$

$$\int\frac{dy}{\sqrt{1-y^2}} = \int\frac{dx}{\sqrt{1-x^2}}$$

$\arcsin y = \arcsin x + c$

$y = \sin(\arcsin x + c).$

$\quad = \sin(\arcsin x)\cos c + \cos(\arcsin x)\sin c$



$y = \arcsin x \Rightarrow \sin y = x$

$\cos y = \cos(\arcsin x)$

$\quad = \dfrac{adj}{hyp}$

$\quad = \dfrac{\sqrt{1-x^2}}{1}$

$= x\cos c + \sqrt{1-x^2}\,\sin c.$

$y(0) = \frac{\sqrt{3}}{2} = 0 + \sin c$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = c = \frac{\pi}{3}.$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

So $y(x) = \frac{1}{2}x + \frac{\sqrt{3}}{2}\sqrt{1-x^2}$

6. $\frac{dy}{dx} = \frac{y^2-1}{x^2-1}$; $y(2)=2$

$\int \frac{dy}{y^2-1} = \int \frac{dx}{x^2-1}$

$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$

$1 = A(y+1) + B(y-1)$

$y=1,\ 2A=1 \Rightarrow A = \frac{1}{2}$

$y=-1,\ -2B=1 \Rightarrow B = -\frac{1}{2}.$

$\int\left(\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\right)dy = \int\left(\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right)dx$

$\frac{1}{2}\left(\ln|y-1| - \ln|y+1|\right) = \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + c$

$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$

$$\frac{y-1}{y+1} = e^{\ln\left|\frac{x-1}{x+1}\right| + c} = e^{\ln\left|\frac{x-1}{x+1}\right|} e^{c} = \left(\frac{x-1}{x+1}\right) c_1$$

$$\frac{2-1}{2+1} = \left(\frac{2-1}{2+1}\right) c_1$$

$$\frac{1}{3} = \frac{1}{3} c_1 \implies c_1 = 1.$$

So $$\frac{y-1}{y+1} = \frac{x-1}{x+1}$$

$$y-1 = (y+1)\left(\frac{x-1}{x+1}\right) = y\left(\frac{x-1}{x+1}\right) + \left(\frac{x-1}{x+1}\right)$$

$$y - y\left(\frac{x-1}{x+1}\right) = 1 + \frac{x-1}{x+1}$$

$$y\left(1 - \frac{x-1}{x+1}\right) = \frac{(x+1)+(x-1)}{x+1} = \frac{2x}{x+1}$$

$$y\left(\frac{(x+1)-(x-1)}{x+1}\right) = \frac{2x}{x+1}$$

$$y\left(\frac{2}{x+1}\right) = \frac{2x}{x+1}.$$

$$y(x) = x.$$

7. Find a singular solution of $(e^x + e^{-x})\frac{dy}{dx} = y^2$

$$\int \frac{dy}{y^2} = \int \frac{dx}{e^x + e^{-x}}$$

$$-y^{-1} = \int \frac{e^x\,dx}{e^{2x}+1}$$

Note: $\frac{d}{du}\left[\arctan u\right] = \frac{u'}{1+u^2}$

$u = e^x \qquad u^2 = e^{2x}$

$du = e^x$

$$-y^{-1} = \arctan(e^x) + c$$

$$y^{-1} = -\arctan(e^x) + c$$

$$y(x) = \frac{1}{-\arctan(e^x) + c}$$