2-3 Linear Equations Examples

Find the general solution of the given DE. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $x\frac{dy}{dx} - y = x^2 \sin x$

$$\frac{dy}{dx} \underbrace{- \frac{1}{x}}_{P(x)} y = \underbrace{x\sin(x)}_{Q(x)}$$

$P(x)$ cont $(-\infty,0) \cup (0,\infty)$

$Q(x)$ cont on $\mathbb{R}$

$I: x>0$

IF: $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1}$

Then $x^{-1}\frac{dy}{dx} - x^{-2}y = \sin(x)$

$$\frac{d}{dx}\left[x^{-1}y\right] = \sin(x)$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int \sin(x)dx$$

$$x^{-1}y = -\cos(x) + C$$

$$y(x) = Cx - x\cos(x)$$

2. $x\,dy - 4(y+x^6)\,dx = 0$

$x\,dy = 4(y+x^6)\,dx$

$\dfrac{dy}{dx} = \dfrac{4(y+x^6)}{x}$ $\qquad$ y cont $\mathbb{R}$

$\qquad\qquad\qquad\qquad$ x cont $\mathbb{R}\setminus\{0\}$.

$\dfrac{dy}{dx} - \underbrace{\dfrac{4}{x}}_{P(x)}y = 4x^5$

$IF: e^{-4\int x^{-1}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$

$x^{-4}\dfrac{dy}{dx} - \dfrac{4}{x^5}y = 4x$

$\dfrac{d}{dx}\left[x^{-4}y\right] = 4x$

$\int \dfrac{d}{dx}\left[x^{-4}y\right]dx = 4\int x\,dx$

$x^{-4}y = 4\left(\tfrac{1}{2}\right)x^2 + c$

$y(x) = 2x^6 + cx^4$. $\quad , x>0$

$\dfrac{\partial f(x,y)}{\partial y} = \dfrac{x(4) - 4(y+x^6)(0)}{x^2} = \dfrac{4}{x}$ $\qquad$ y cont every where

x: cont every where

except $x=0$.

3. Solve the IVP $x\frac{dy}{dx} - y = 2x^2$, $y(1) = 5$.

Give the largest interval over which the solution is defined.

$\frac{dy}{dx} \underbrace{-\frac{1}{x}}_{P(x)} y = \underbrace{2x}_{Q(x)}$

$P(x)$: cont on $(-\infty,0)\cup(0,\infty)$

$Q(x)$: cont on $\mathbb{R}$

Common: $(-\infty,0)\cup(0,\infty)$

Note: 1 is in $(0,\infty)$ so

$I = (0,\infty)$.

IF: $e^{-\int x^{-1}dx} = e^{-\ln x}$

$= e^{\ln x^{-1}}$

$= x^{-1}$

$x^{-1}\left(\frac{dy}{dx} - \frac{y}{x} = 2x\right)$

$x^{-1}\frac{dy}{dx} - \frac{y}{x^2} = 2$

$\frac{d}{dx}\left[x^{-1}y\right] = 2$

$\int \frac{d}{dx}\left[x^{-1}y\right]dx = 2\int dx$

$x^{-1}y = 2x + C$

$y(x) = 2x^2 + Cx$

$y(1) = 5 = 2(1^2) + C$

$C + 2 = 5 \Rightarrow C = 3$

$y(x) = 2x^2 + 3x.$

4. Solve the IVP $\frac{dy}{dx} + y = f(x)$, $y(0) = 1$

where $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1. \end{cases}$ just as in Ex 6.

For $0 \le x \le 1$

$$\frac{dy}{dx} + \underbrace{1}_{P(x)}y = e^x$$

IF $e^{\int dx} = e^x$

$$e^x\left(\frac{dy}{dx} + y = 1\right)$$

$$e^x\frac{dy}{dx} + e^x y = e^x$$

$$\frac{d}{dx}\left[e^x y\right] = e^x$$

$$\int \frac{d}{dx}\left[e^x y\right] dx = \int e^x dx$$

$$ye^x = e^x + c_1$$

$$y(x) = c_1 e^{-x} + 1$$

$$y(0) = 1 = c_1 + 1 \Rightarrow c_1 = 0$$

$$y(x) = 1$$

For $x > 1$

$$\frac{dy}{dx} + y = -1$$

IF: e^x

$$e^x\left(\frac{dy}{dx} + y = -1\right)$$

$$e^x\frac{dy}{dx} + e^x y = -e^x$$

$$\frac{d}{dx}\left[e^x y\right] = -e^x$$

$$\int \frac{d}{dx}\left[e^x y\right] dx = -\int e^x dx$$

$$ye^x = -e^x + c_2$$

$$y(x) = c_2 e^{-x} - 1$$

Now $y(x) = \begin{cases} 1, & 0 \le x \le 1 \\ c_2 e^{-x} - 1, & x > 1 \end{cases}$ Need $y(x)$ to be cont at $x = 1$

ie $\lim_{x \to 1^+} y(x) = y(1)$

So $c_2 e^{-1} - 1 = 1$

$c_2 e^{-1} = 2$

$c_2 = 2e$

$$\Longrightarrow y(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2e^{-x+1} - 1, & x > 1 \end{cases}$$