3.1 Linear Models

1. Newton's Law of Cooling: the rate at which an object cools is proportional to the difference between the object's temperature (T) and the temperature of the surrounding medimum (T_m), ie

$$\frac{dT}{dt} = K(T - T_m)$$ where K is a constant of proportionality.

Suppose a cold beer at 40°F is placed into a warm room at 70°F. Suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton's law of cooling to find the temperature 25 minutes after the beer was placed in the room.

First we need our initial value problem (IVP).

$$\frac{dT}{dt} = K(T - 70)$$

$$\begin{cases} T(0) = 40 \\ T(10) = 48 \end{cases}$$

ALWAYS write the initial conditions beside the DE, otherwise you do not have an IVP.

Note: This is a first order, linear, separable equation, so

$$\int \frac{dT}{T-70} = \int K\,dt$$

$$\ln|T-70| = Kt + c$$

$$T(t) - 70 = e^{Kt+c}$$

$$T(t) = 70 + c_1 e^{Kt}$$

This general solution is said to be a two parameter family of solutions since we have two arbitrary constants c_1, K

Since we need a particular solution to find the temperature of the beer after 25 minutes, we need to know what c_1 and k are. Hence the reason for two initial conditions.

$$T(0) = 40 = 70 + c_1 e^{k(0)}$$

$$\Rightarrow c_1 = 40 - 70$$

$$= -30$$

So $T(t) = 70 - 30e^{kt}$

Now $T(10) = 48 = 70 - 30e^{10k}$

$$-22 = -30e^{10k}$$

$$e^{10k} = \frac{11}{15}$$

$$k = \frac{1}{10}\ln\left(\frac{11}{15}\right)$$

$$\approx -0.0310$$

So $T(t) = 70 - 30\,e^{-0.0310t}$

Now $T(25) = 70 - 30e^{-0.0310(25)}$

$$\approx 56.18°F.$$

2. Growth and Decay: The rate at which a population grows/declines at a certain time is proportional to the total population at that time.

ie $\frac{dx}{dt} = kx$ where k is a growth/decay constant

A biologist starts with 100 cells in a culture. After 1 day, he counts 300. What is the reproductive rate? What will be the number of cells after 5 days?

Let $P(t)$ = population of cells at time t (in days).

Then the IVP is $\frac{dP}{dt} = kP$

$$\begin{cases} P(0) = 100 \\ P(1) = 300 \end{cases}$$

Again, this is a first order, linear, separable equation

So $\int \frac{dP}{P} = \int k\,dt$

$$\ln|P| = kt + c$$

$$P(t) = e^{kt+c}$$

$$= e^{kt}e^{c}$$

$$= c_1 e^{kt}$$

Then $P(0) = 100 = c_1 e^{0(k)}$

$$\Rightarrow c_1 = 100$$

So $P(t) = 100e^{kt}$

$$P(1) = 300 = 100e^{k}$$

$$3 = e^{k}$$

$$k = \ln(3)$$

$$\approx 1.0986$$

giving $P(t) = 100e^{1.0986t}$

Then $P(5) = 100e^{1.0986(5)}$

$$\approx 24{,}298.5$$

3. Personal Finance: The rate at which an investment grows (declines) at a given time is equal to the sum of the rate at which interest accrues and the deposits (withdrawls) made to the account.

ie $\frac{dS}{dt} = rS + K$ where $S(t)$ = savings at time t
r = interest rate
K = deposits/withdrawls
assumed to be constant.

Suppose you have just started work and have no initial savings. If you decide to save \$2000 each year with an interest rate of 5% per year compounded continuously, how much will you accumulate after 30 years?

$$\frac{dS}{dt} = 0.05S + 2000$$
$$S(0) = 0$$

Note: this is a first order, linear, nonseparable equation
Rewriting the DE in standard form, we have

$$S' - 0.05S = 2000$$
$$IF = e^{\int(-0.05)dt} = e^{-0.05t}$$
$$e^{-0.05t}\left[S' - 0.05S = 2000\right]$$
$$\frac{d}{dt}\left[e^{-0.05t}S\right] = 2000e^{-0.05t}$$
$$\int\frac{d}{dt}\left[e^{-0.05t}S\right]dt = 2000\int e^{-0.05t}$$

$e^{-0.05t}S = -40000e^{-0.05t} + C$

$S(t) = -40000 + Ce^{0.05t}$

$S(0) = 0 = -40000 + Ce^{0}$

$\Rightarrow C = 40000$

So $S(t) = 40000(e^{0.05t} - 1)$

Then $S(30) = 40000(e^{0.05(30)} - 1)$

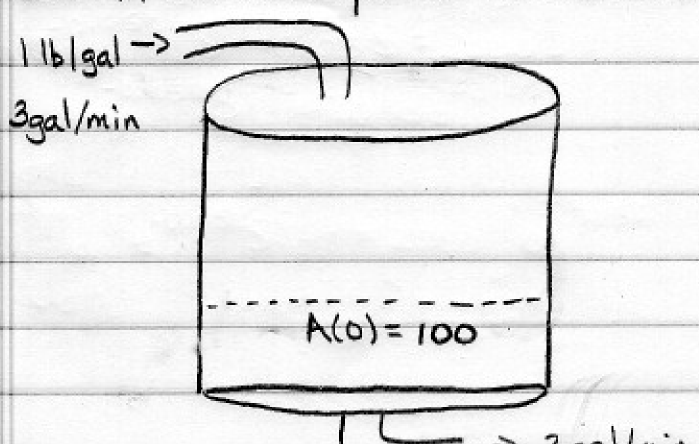
$\approx \$139{,}267.56$

4. <u>Mixture of 2 Solutions</u>: The rate of change in the amount of a substance in a well mixed solution is equal to the input rate of that substance minus the output rate.

$$\frac{dA}{dt} = \underset{R_i}{\text{input}} - \underset{R_o}{\text{output}}$$

where rate of input/output = concentration · flow.

A tank with a capacity of 500 gal contains 200 gal of H_2O with 100 lb NaCl in solution. H_2O containing 1 lb/gal of salt enters the tank at a rate 3 gal/min. The well mixed solution is then pumped out at a rate of 2 gal/min. Find the amount of NaCl in the tank prior to the tank overflowing.



R_i = (concentration)(flow)

$= \left(\frac{1\,lb}{gal}\right)\left(\frac{3\,gal}{min}\right)$

$= 3\,lb/min$

R_o = (concentration)(flow)

$= \left(\frac{A(t)}{200+(3-2)t}\,\frac{lb}{gal}\right)\left(\frac{2\,gal}{min}\right)$

$= \frac{2A(t)}{200+t}\,\frac{lb}{gal}$

Note: tank is filling up by a gallon per minute

So we have $\frac{dA}{dt} = 3 - \frac{2A}{200+t}$ $\quad t \in [0, 300]$

$A(0) = 100$

Again, this is a first order, linear, nonseparable equation only this time we have a variable coefficient.

Putting the equation in standard form, we have

$$\frac{dA}{dt} + \left(\frac{2}{200+t}\right)A = 3$$

IF: $e^{2\int \frac{dt}{200+t}} = e^{2\ln(200+t)} = e^{\ln(200+t)^2} = (200+t)^2$

Then $(200+t)^2\left[\frac{dA}{dt} + \frac{2A}{200+t} = 3\right]$

$$\frac{d}{dt}\left[(200+t)^2 A\right] = 3(200+t)^2$$

$$\int \frac{d}{dt}\left[(200+t)^2 A\right]dt = 3\int (200+t)^2\,dt$$

$$(200+t)^2 A = 3\left[\frac{(200+t)^3}{3}\right] + C$$

$$A(t) = 200 + t + \frac{C}{(200+t)^2}$$

$$A(0) = 100 = 200 + 0 + \frac{C}{(200)^2}$$

$$-100 = \frac{C}{(200)^2}$$

$$\Rightarrow C = -100(200)^2 = -4{,}000{,}000$$

So $A(t) = 200 + t - \dfrac{4{,}000{,}000}{(200+t)^2}$ for $t \in [0, 300]$.