3.2 Nonlinear Models

1. <u>Logistic Equation</u>: The number $N(t)$ of supermarkets throughout the country that use a computerized checkout system is described by the IVP

$$\frac{dN}{dt} = N(1 - 0.0005N)$$

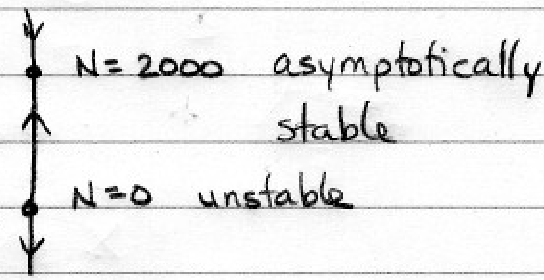
$$N(0) = 1$$

a. Use the phase portrait concept to predict how many supermarkets are expected to adopt the new procedure over a long period of time. Sketch a solution curve.

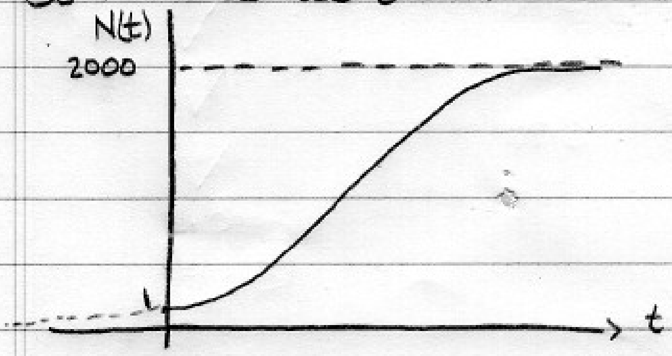
$$\frac{dN}{dt} = N(1 - 0.0005N) = 0$$

$$\Rightarrow N = 0, 2000$$

Interval	TV	+/-	↑/↓
$(-\infty, 0)$	-1	$-$	↓
$(0, 2000)$	1	$+$	↑
$(2000, \infty)$	2001	$-$	↓



So $N = 2000$ as $t \to \infty$.



b. Solve the IVP. How many supermarkets are expected to adopt the new technology when $t=10$?

$$\int \frac{dN}{N(1-0.0005N)} = \int dt$$

Partial Fractions: $\frac{1}{N(1-0.0005N)} = \frac{A}{N} + \frac{B}{1-0.0005N}$

$1 = A(1-0.0005N) + BN$

$N=0 \Rightarrow A=1$

$N=2000 \Rightarrow 1 = 2000B$

$\Rightarrow B = \frac{1}{2000} = 0.005$

$$\int \left(\frac{1}{N} + \frac{0.005}{1-0.005N}\right) dN = \int dt$$

$$\int \left(\frac{1}{N} + \frac{1}{2000-N}\right) dN = \int dt$$

$\ln|N| - \ln|2000-N| = t+c$

$\ln\left|\frac{N}{2000-N}\right| = t+c$

$\frac{N}{2000-N} = e^{t+c} = c_1 e^t$

$N = (2000-N)c_1e^t = 2000c_1e^t - Nc_1e^t$

$N(1+c_1e^t) = 2000c_1e^t$

$N(t) = \frac{2000c_1e^t}{1+c_1e^t}$

$$N(0) = 1 = \frac{2000\,c_1}{1+c_1}$$

$$1 + c_1 = 2000\,c_1$$

$$1 = 1999\,c_1$$

$$c_1 = \frac{1}{1999}$$

So $N(t) = \dfrac{\frac{2000}{1999}e^t}{1 + \frac{1}{1999}e^t} = \dfrac{1.0005e^t}{1+0.0005e^t} = \dfrac{1.0005}{0.0005+e^{-t}}$

Then $N(10) = \dfrac{1.0005}{0.0005+e^{-10}} = 1834$

2. Chemical Reactions: Two chemicals A and B are combined to form a chemical C. The rate of the reaction is proportional to the product of instantaneous amounts of A and B not converted to C. Initially, there are 30 grams of A and B each, and for every 2 grams of B, 3 grams of A is used. It is observed after 10 minutes that 5 grams of C is formed. How much is formed in 20 minutes? What is the limiting amount of C after a long time? How much of A and B remain after a long time?

Let $X(t)$ = the amount of C at time t.

Then $\dfrac{dX}{dt} = K\left(a - \dfrac{M}{M+N}X\right)\left(b - \dfrac{N}{M+N}X\right)$ where $a = b = 30$

$M = 3$

$N = 2$

$$\frac{dX}{dt} = K\left(30 - \frac{3}{5}x\right)\left(30 - \frac{2}{5}x\right)$$

$$= K_1(50 - X)(75 - x) \qquad \text{where } K_1 = \frac{6}{25}K$$

We also have the boundary conditions

$$\begin{cases} X(0) = 0 \\ X(10) = 5 \end{cases}$$

<u>Note</u>: This is a 1st order, nonlinear, separable, autonomous equation.

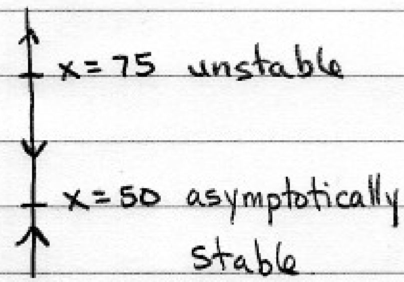
To find the limiting amount of C, we can use the fact that this an autonomous equation.

$$\frac{dX}{dt} = K_1(50 - X)(75 - X) = 0$$

Critical points: $X = 50, 75$

Int	TV	+/-	↑/↓
$(-\infty, 50)$	0	+	↑
$(50, 75)$	60	-	↓
$(75, \infty)$	100	+	↑

Phase line



So the limiting amount of C is 50 grams

To find how much of A and B we have remaining, we take

A: $30 - \frac{3}{5}(50) = 0$ grams of A

B: $30 - \frac{2}{5}(50) = 10$ grams of B.

To find how much of C we have after 20 mins, we must first solve for X(t) by separation of variables.

$$\int \frac{dX}{(50-X)(75-X)} = K_1 \int dt$$

Now by partial fractions,

$$\frac{1}{(50-X)(75-X)} = \frac{A}{50-X} + \frac{B}{75-X}$$

$$\Rightarrow 1 = A(75-X) + B(50-X)$$

$$X = 50, \quad 1 = 25A$$

$$\text{So } A = \frac{1}{25}$$

$$X = 75, \quad 1 = -25B$$

$$\text{So } B = -\frac{1}{25}$$

$$\text{So } \frac{1}{25}\int\left(\frac{1}{50-X} - \frac{1}{75-X}\right)dx = K_1\int dt$$

$$\frac{1}{25}\Big[-\ln|50-X| - (-1)\ln|75-X|\Big] = K_1 t + C$$

$$\ln|75-X| - \ln|50-X| = K_2 t + C_1 \qquad K_2 = 25K_1,\ C_1 = 25C_2$$

$$\ln\left|\frac{75-X}{50-X}\right| = K_2 t + C_1$$

$$\frac{75-X}{50-X} = e^{K_2 t + C_1} = e^{K_2 t}e^{C_1} = C_2 e^{K_2 t}$$

$$75 - X = (50-X)C_2 e^{K_2 t} = 50C_2 e^{K_2 t} - XC_2 e^{K_2 t}$$

$$c_2 x e^{K_2 t} - x = 50 c_2 e^{K_2 t} - 75$$

$$x(c_2 e^{K_2 t} - 1) = 50 c_2 e^{K_2 t} - 75$$

So $x(t) = \dfrac{50 c_2 e^{K_2 t} - 75}{c_2 e^{K_2 t} - 1}$

$$x(0) = 0 = \frac{50 c_2 - 75}{c_2 - 1}$$

$$50 c_2 = 75$$

$$\Rightarrow c_2 = \frac{3}{2}$$

Now $x(t) = \dfrac{50(3/2) e^{K_2 t} - 75}{(3/2) e^{K_2 t} - 1} = \dfrac{75 e^{K_2 t} - 75}{\frac{3}{2} e^{K_2 t} - 1} = \dfrac{150(e^{K_2 t} - 1)}{3 e^{K_2 t} - 2}$

$$x(10) = 5 = \frac{150(e^{10K_2} - 1)}{3e^{10K_2} - 2}$$

$$5(3^{10K_2} - 2) = 150 e^{10K_2} - 150$$

$$150 e^{10K_2} - 15 e^{10K_2} = 150 - 10 = 140$$

$$135 e^{10K_2} = 140$$

$$K_2 = \frac{1}{10} \ln\left(\frac{140}{135}\right) = \frac{1}{10} \ln\left(\frac{28}{27}\right) \approx 0.003637$$

So $x(t) = \dfrac{150(e^{0.003637t} - 1)}{3e^{0.003637t} - 2}$

Now $x(20) = \dfrac{150(e^{0.003637(20)} - 1)}{3e^{0.003637(20)} - 2} \approx 11.19$ grams.

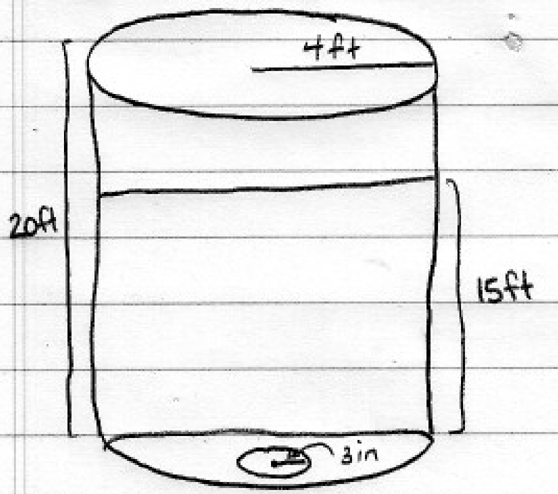
<u>Note</u>: $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} \dfrac{150(e^{0.003637t}-1)}{3e^{0.003637t}-2} = \dfrac{\infty}{\infty}$, indeterminant

rewriting $x(t)$ as $x(t) = \dfrac{150(1-e^{-0.003637t})}{3-2e^{-0.003637t}}$

we get $\lim\limits_{t\to\infty} x(t) = \dfrac{150}{3} = 50$, the same limiting value for C

we found before using our phase line.

3. <u>Draining Tank</u>: A tank in the form of a right circular cylinder standing on end is leaking water through a circular hole in the bottom. When friction and contraction of water at the hole are ignored, the height h of the water in the tank is described by
$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$
where A_w and A_h are the cross-sectional areas of the water and the hole, respectively. Now suppose that the tank is 20 feet high and has a radius of 4 feet with a circular hole of 3 inches. If the tank is initially three quarters full, how long will it take to empty? Assume $g = 32\ ft/s^2$.



$A_w = \pi r^2 = \pi(4)^2 = 16\pi$

$A_h = \pi r^2 = \pi\left(\dfrac{3}{12}\right)^2 = \dfrac{\pi}{16}$

$h(0) = \dfrac{3}{4}(20) = 15.$

Now $\frac{dh}{dt} = \frac{-\pi}{\frac{16}{16\pi}}\sqrt{2(32)h} = \frac{-\pi}{16(16\pi)}(8\sqrt{h}) = \frac{-1}{32}\sqrt{h}$

$h(0) = 15.$

<u>Note</u>: This is a 1st order, nonlinear, <u>separable</u>, autonomous equation.

Separating the equation gives

$$\int \frac{dh}{\sqrt{h}} = -\int \frac{1}{32}\,dt$$

$$2\sqrt{h} = \frac{-t}{32} + c$$

$$\sqrt{h} = \frac{-t}{64} + c_1 \qquad \text{where } c_1 = \frac{1}{2}c$$

So $h(t) = \left(\frac{-t}{64} + c_1\right)^2$

Now $h(0) = (0 + c_1)^2 = 15$

$\Rightarrow c_1 = \sqrt{15}$

Then $h(t) = \left(\frac{-t}{64} + \sqrt{15}\right)^2$

Let t_0 be the time the tank is completely empty.

$h(t_0) = \left(\frac{-t_0}{64} + \sqrt{15}\right)^2 = 0$

$\Rightarrow t_0 = 64\sqrt{15} \approx 247.871$ s

≈ 4.13 min.