4.2: Reduction of Order

The indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order to find a second linearly independent solution $y_2(x)$.

1. $y''+2y'+y=0$; $y_1(x)=xe^{-x}$

This is 2nd order, linear, homogeneous equation with constant coefficients

Assume $y_2(x)=u(x)y_1(x)=uxe^{-x}$

$$y_2'=u'xe^{-x}+u(xe^{-x})'$$
$$=u'xe^{-x}+ue^{-x}-uxe^{-x}$$
$$y_2''=u''xe^{-x}+u'(e^{-x}-xe^{-x})+u'e^{-x}-ue^{-x}-u'xe^{-x}-u(e^{-x}-xe^{-x})$$
$$=u''xe^{-x}+u'(e^{-x}+e^{-x}-xe^{-x}-xe^{-x})+u[-e^{-x}-e^{-x}+xe^{-x}]$$
$$=u''xe^{-x}+2u'e^{-x}-2u'xe^{-x}-2ue^{-x}+uxe^{-x}$$

Now plugging y_2 into the DE

$$u''xe^{-x}+2u'e^{-x}-2u'xe^{-x}-2ue^{-x}+uxe^{-x}+2(u'xe^{-x}+ue^{-x}-uxe^{-x})+uxe^{-x}=0$$
$$u''xe^{-x}+u'[\underbrace{2e^{-x}-2xe^{-x}+2xe^{-x}}_{2e^{-x}}]+u[\underbrace{-2e^{-x}+xe^{-x}+2e^{-x}-2xe^{-x}+xe^{-x}}_{0}]=0$$

$$u''xe^{-x}+2e^{-x}u'=0$$
$$xu''+2u'=0$$

Let $w=u'$
$$w'=u''$$

$\Rightarrow xw'+2w=0$ Note: this is a 1st order, linear, separable equation

$$x\frac{dw}{dx}=-2w$$
$$\int\frac{dw}{w}=\int-\frac{2}{x}\,dx$$

$\ln|w| = -2\ln|x| + c = \ln|x^{-2}| + c$,

So $w = e^{\ln|x^{-2}| + c}$

$= e^{\ln|x^{-2}|} e^{c}$

$= c_1 e^{\ln|x^{-2}|}$

$= c_1 x^{-2}$

But $w = u'$

So $u(x) = \int c_1 x^{-2} dx$

$= -c_1 x^{-1} + c_2$

$= \tilde{c}_1 x^{-1} + c_2$

Let $\tilde{c}_1 = 1$

$c_2 = 0$.

Then $u(x) = x^{-1}$

So $y_2(x) = u(x) y_1(x)$

$= (x^{-1})(xe^{-x})$

$= e^{-x}$

Now our general solution is $y(x) = c_1 xe^{-x} + c_2 e^{-x}$

Note: If we leave $u(x) = \tilde{c}_1 x^{-1} + c_2$, when we multiply by xe^{-x} we get $u(x) y_1(x) = \tilde{c}_1 e^{-x} + c_2 xe^{-x}$. However, this second term is just a scalar multiple of $y_1(x)$, so it will be "absorbed." This allows us to treat c_2 as 0. Then by the Superposition Principle, any scalar multiple of a solution is also a solution, so we can treat $\tilde{c}_1$ as 1, giving us $u(x) = x^{-1}$.

2. $y''+9y=0$; $y_1(x)=\sin(3x)$

This is a 2nd order, linear, homogeneous equation with constant coefficients

Assume $y_2(x)=u(x)y_1(x)$
$=u\sin(3x)$

$y_2'=u'\sin(3x)+3u\cos(3x)$

$y_2''=u''\sin(3x)+3u'\cos(3x)+3u'\cos(3x)-9u\sin(3x)$
$=u''\sin(3x)+6u'\cos(3x)-9u\sin(3x)$

Now plugging y_2 into the equation

$$u''\sin(3x)+6u'\cos(3x)\underbrace{-9u\sin(3x)+9u\sin(3x)}_{0}=0$$

$u''\sin(3x)+6u'\cos(3x)=0$

Let $w=u'$
$w'=u''$

$\sin(3x)w'+6\cos(3x)w=0$

Note: this is a 1st order, linear, separable equation

$$\sin(3x)\frac{dw}{dx}=-6\cos(3x)w$$

$$\int\frac{dw}{w}=-6\int\frac{\cos(3x)}{\sin(3x)}dx$$

Let $v=\sin(3x)$
$dv=3\cos(3x)dx$

$$=-2\int\frac{3\cos(3x)}{\sin(3x)}dx$$

$=-2\ln|\sin(3x)|$

So $\ln|w|=\ln|(\sin(3x))^{-2}|=\ln|\csc^2(3x)|$

$w=\csc^2(3x)$

But $w = u'$

So $u(x) = \int \csc^2(3x)\,dx$

$$= \frac{1}{3}\cot(3x)$$

$\Rightarrow u(x) = \cot(3x)$

Then $y_2(x) = u(x)y_1(x)$

$$= \cot(3x)\sin(3x)$$

$$= \frac{\cos(3x)}{\sin(3x)}\sin(3x)$$

$$= \cos(3x).$$

Now the general solution is $y(x) = c_1\sin(3x) + c_2\cos(3x)$.

3. $x^2y'' + 2xy' - 6y = 0$; $y_1(x) = x^2$

This is a 2nd order, linear, homogeneous equation with variable coefficients. Assume $x > 0$.

Assume $y_2(x) = u(x)y_1(x)$

$$= ux^2$$

$y_2' = u'x^2 + 2ux$

$y_2'' = u''x^2 + 2u'x + 2u'x + 2u$

$$= u''x^2 + 4u'x + 2u$$

Now plugging y_2 into the DE, we get

$x^2(u''x^2 + 4u'x + 2u) + 2x(u'x^2 + 2ux) - 6ux^2 = 0$

$u''x^4 + u'[\underbrace{4x^3 + 2x^3}_{6x^3}] + u[\underbrace{2x^2 + 4x^2 - 6x^2}_{0}] = 0$

$u''x^4 + 6u'x^3 = 0$

$u'' + \frac{6}{x}u = 0$

Let $w = u'$

$w' = u''$

$\Rightarrow \frac{dw}{dx} + \frac{6}{x} w = 0.$ This is a 1st order, linear, separable equation

$\frac{dw}{dx} = -\frac{6}{x} w$

$\int \frac{dw}{w} = -6 \int \frac{dx}{x}.$

$\ln|w| = -6\ln|x|.$

$= \ln x^{-6}$

$\Rightarrow w = e^{\ln x^{-6}} = x^{-6}$

Then $w = u'$, so $u(x) = \int x^{-6}\,dx$

$= -\frac{1}{6} x^{-5}$

$\Rightarrow u(x) = x^{-5}$

So $y_2(x) = u(x)\, y_1(x)$

$= x^{-5} x^{2}$

$= x^{-3}$

Now our general solution is $y(x) = c_1 x^2 + c_2 x^{-3}$

4. $(x-1)y''-xy'+y=0\ ;\ y_1(x)=e^x$

This is a 2nd order, linear, homogeneous equation with variable coefficients

Assume $y_2(x)=u(x)y_1(x)$
$=ue^x$

$y_2'=u'e^x+ue^x$

$y_2''=u''e^x+2u'e^x+ue^x$

Then plugging y_2 into the DE, we get

$(x-1)(u''e^x+2u'e^x+ue^x)-x(u'e^x+ue^x)+ue^x=0$

$$(x-1)u''+(\underbrace{2(x-1)-x}_{x-2})u'+(\underbrace{(x-1)-x+1}_{0})u=0$$

$(x-1)u''+(x-2)u'=0.$

Let $w=u'$

$w'=u'$

$\Rightarrow (x-1)\dfrac{dw}{dx}=-(x-2)w$

This is a 1st order, linear, separable equation

$$\int\frac{dw}{w}=-\int\left(\frac{x-2}{x-1}\right)dx$$
$$=-\int\left(1-\frac{1}{x-1}\right)dx$$

$\ln|w|=-(x-\ln|x-1|)$
$=-x+\ln|x-1|$

$\Rightarrow w=e^{-x+\ln|x-1|}$
$=e^{-x}e^{\ln|x-1|}$
$=e^{-x}(x-1)$

Then $w = u' \Rightarrow u(x) = \int e^{-x}(x-1)dx$

$s = x-1 \qquad dt = e^{-x}$

$ds = dx \qquad t = -e^{-x}$

$= -(x-1)e^{-x} - \int(-e^{-x})dx$

$= -(x-1)e^{-x} - e^{-x}$

$= -xe^{-x}$

So $y_2(x) = u(x)y_1(x)$

$= e^x(-xe^{-x})$

$= -x$

$\Rightarrow y_2(x) = x$

Then our general solution is $y(x) = c_1 e^x + c_2 x$