4.5 Method of Undetermined Coefficients.

Give the <u>smallest</u> linear differential operator that annihilates the given function.

<u>Method</u>: Find the roots that correspond to each term. Then derive the appropriate characteristic equation to get these terms. This will be your annihilator once you replace the r's with D's.

1. $a(x) = \underbrace{1+6x-2x^3}_{r=0 \text{ multiplicity } 4.}$

$r^4 = 0$

So $D^4[a(x)] = 0$.

2. $b(x) = 1 + 7e^{2x}$

$1 \to r = 0$

$7e^{2x} \to r = 2$.

So the characteristic eq to get these terms is $r(r-2) = 0$

$\Rightarrow D(D-2)[b(x)] = 0$.

3. $c(x) = e^{x}\cos(2x) + 2e^{-2x}\sin(3x)$

$e^{x}\cos(2x) \to r = -1 \pm 2i$

$(r-1)^2 = -4$

$(r-1)^2 + 4 = 0$

$2e^{-2x}\sin(3x) \to r = -2 \pm 3i$

$(r+2)^2 = -9$

$(r+2)^2 + 9 = 0$

So the characteristic eq is $((r-1)^2+4)((r+2)^2+9) = 0$

$\Rightarrow [(D-1)^2+4][(D+2)^2+9][c(x)] = 0$

4. $d(x) = 7x\cos(3x) + \cos(3x)$

$7x\cos(3x) \rightarrow r = \pm 3i, \pm 3i$

$\cos(3x) \rightarrow r = \pm 3i$

So the characteristic eq to get these terms is

$(r^2+9)^2 = 0.$

$\Rightarrow (D^2+9)^2[d(x)] = 0.$

<u>Note</u>: $(D^2+9)^3$ will annihilate $d(x)$ since

$$\begin{aligned}(D^2+9)^3[d(x)] &= (D^2+9)(D^2+9)^2[d(x)] \\ &= (D^2+9)[0] \\ &= 0.\end{aligned}$$

But we want the <u>smallest</u> annihilator that kills $d(x)$. Otherwise, we'll end up with extra terms in y_p that will "zero" out on us once we plug y_p in the DE and match coefficients.

5. $e(x) = x^2 + e^{-x} + e^{-x}\sin(4x).$

$x^2 \rightarrow r = 0$ multiplicity of 3.

$e^{-x} \rightarrow r = 1$

$e^{-x}\sin(4x) \rightarrow r = -1 \pm 4i \Rightarrow (r+1)^2 + 16 = 0.$

Characteristic eq: $r^3(r-1)((r+1)^2+16) = 0$

$\Rightarrow D^3(D-1)((D+1)^2+16)[e(x)] = 0.$

When can we use Method of Undetermined Coefficients?

In the 2nd order case, we have $ay''+by'+cy=g(x)$ st.

1. The DE is <u>linear</u> with constant coefficients on y's.
2. Each term of $g(x)$ corresponds to something we can get from the characteristic eq $ar^2+br+c=0$.

$r=0$ mult $n \Rightarrow c_1+c_2x+c_3x^2+\dots+c_nx^{n-1}$

$r=\alpha$ mult $n \Rightarrow c_1e^{\alpha x}+c_2xe^{\alpha x}+\dots+c_nx^{n-1}e^{\alpha x}$

$r=\alpha\pm\beta i$ mult $n \Rightarrow \begin{cases} a_1e^{\alpha x}\cos(\beta x)+a_2xe^{\alpha x}\cos(\beta x)+\dots+a_nx^{n-1}e^{\alpha x}\cos(\beta x) \\ b_1e^{\alpha x}\sin(\beta x)+b_2xe^{\alpha x}\sin(\beta x)+\dots+b_nx^{n-1}e^{\alpha x}\sin(\beta x). \end{cases}$

(Incidently, $g(x)$ will be continuous on any I since the functions above are continuous everywhere. Continuity of $g(x)$ will be important when we look at Variation of Parameters).

Solve the given DE using MUC.

6. $y''-2y'+y=x^3+4x$

First solve for $y''-2y'+y=0$.

Now assume $y(x)=e^{rx}$ satisfies the eq.

$y'(x)=re^{rx}$

$y''(x)=r^2e^{rx}$

$r^2e^{rx}-2re^{rx}+e^{rx}=0$

$e^{rx}(r^2-2r+1)=0$

$r^2-2r+1=0$ since $e^{rx}\neq 0$

$(r-1)^2=0$

$\Rightarrow r=1,1.$

Then $y_1(x) = e^x$

$y_2(x) = xe^x$ using Reduction of Order (4.2).

So $y_h(x) = c_1e^x + c_2xe^x$

Now we look at $g(x) = x^3 + 4x$

$x^3 \to r = 0$ mult 4

$4x \to r = 0$ mult 2.

$\therefore$ the characteristic eq to get terms of $g(x)$ is $r^4 = 0$

So $D^4(x^3+4x) = 0$

Rewriting the original DE in differential operator notation, we have.

$$(D^2 - 2D + 1)y = x^3 + 4x.$$

Now applying the annihilator of $g(x)$ to both sides

$D^4(D^2 - 2D + 1)y = D^4(x^3 + 4x) = 0$

we have a 6th order linear homogeneous eq with constant coefficients.

In essence, what the annihilator allows us to do is move the roots associated with $g(x)$ to the LHS of the eq.

Again, assume $y(x) = e^{rx}$

Recall that the differential operator on y will factor the same way as the characteristic eq of the DE, we have

$$r^4(r^2 - 2r + 1) = r^4(r-1)^2 = 0.$$

So the roots of this higher order DE are $r = 0$ mult 4

$r = 1$ mult 2.

Moving right to left, we have

$$y(x) = \underbrace{c_1 e^x + c_2 x e^x}_{y_h} + \underbrace{c_3 + c_4 x + c_5 x^2 + c_6 x^3}_{y_p}$$

Recall that y_p is a <u>particular solution</u> when y_p satisfies the RHS of the DE (ie $y_p'' - 2y_p' + y_p = x^3 + 4x$).

$\Rightarrow$ y_p has no parameters (ie we need to find the coefficients of y_p).

To find these coefficients, we will plug y_p into the DE and match coefficients to get c_3 through c_6.

$y_p(x) = A + Bx + Cx^2 + Ex^3$

$y_p'(x) = B + 2Cx + 3Ex^2$

$y_p''(x) = 2C + 6Ex$

So $y_p'' - 2y_p' + y_p = (E)x^3 + (-6E + C)x^2 + (6E - 4C + B)x + (2C - 2B + A)$

$\qquad = x^3 + 0x^2 + 4x + 0$

Matching coefficients, $E = 1$

$-6E + C = 0 \quad \Rightarrow C = 6$

$6E - 4C + B = 4 \quad \Rightarrow B = 22$

$2C - 2B + A = 0 \quad \Rightarrow A = 32.$

So $y_p(x) = 32 + 22x + 6x^2 + x^3$

Now the general solution is

$y(x) = c_1 e^x + c_2 x e^x + 32 + 22x + 6x^2 + x^3$

<u>Note</u>: In order to find c_1, c_2 we must first have the general solution before we can use any given initial conditions.

ie $\begin{cases} y(x_0) = a \\ y'(x_0) = b \end{cases}$ means $\begin{cases} (y_h + y_p)(x_0) = a \\ (y_h + y_p)'(x_0) = b \end{cases}$

7. $y''' - 3y'' + 3y' - y = e^x - x + 16.$

First find $y_h(x)$

$y''' - 3y'' + 3y' - y = 0$

Assume $y(x) = e^{rx}$

$y'(x) = re^{rx}$

$y''(x) = r^2e^{rx}$

$y'''(x) = r^3e^{rx}$

$r^3e^{rx} - 3r^2e^{rx} + 3re^{rx} - e^{rx} = 0$

$e^{rx}(r^3 - 3r^2 + 3r - 1) = 0$

$r^3 - 3r^2 + 3r - 1 = 0$

$(r^3 - 1) - (3r^2 - 3r) = 0$

$(r-1)(r^2 + r + 1) - 3r(r-1) = 0$

$(r-1)(r^2 + r + 1 - 3r) = 0$

$(r-1)(r^2 - 2r + 1) = 0$

$(r-1)(r-1)^2 = 0$

$(r-1)^3 = 0$

$r = 1$ mult 3.

So $y_h(x) = c_1e^x + c_2xe^x + c_3x^2e^x$

Now find the annihilator the RHS $g(x) = e^x - x + 16.$

$e^x \rightarrow r = 1$

$x \rightarrow r = 0$ mult 2.

$16 \rightarrow r = 0$

Characteristic Eq: $r^2(r-1) = 0.$

So $D^2(D-1)[g(x)] = 0$

$$(D^3-3D^2+3D-1)y = e^x - x + 16$$

$$\Rightarrow D^2(D-1)(D^3-3D^2+3D-1)\,y = D^2(D-1)[e^x - x + 16] = 0$$

$$D^2(D-1)(D-1)^3 y = 0$$

Assume $y(x) = e^{rx}$

$$\Rightarrow r^2(r-1)(r-1)^3 = 0$$

$r=0$ mult 2

$r=1$ mult 4

$$y(x) = \underbrace{c_1 e^x + c_2 x e^x + c_3 x^2 e^x}_{y_h} + \underbrace{c_4 x^3 e^x + c_5 + c_6 x}_{y_p}$$

$$y_p(x) = Ax^3e^x + B + Cx$$

$$y_p'(x) = 3Ax^2e^x + Ax^3e^x + C$$

$$y_p''(x) = 6Axe^x + 3Ax^2e^x + 3Ax^2e^x + Ax^3e^x$$

$$= 6Axe^x + 6Ax^2e^x + Ax^3e^x.$$

$$y_p'''(x) = 6Ae^x + 6Axe^x + 12Axe^x + 6Ax^2e^x + 3Ax^2e^x + Ax^3e^x$$

$$= 6Ae^x + 18Axe^x + 9Ax^2e^x + Ax^3e^x.$$

$$y_p''' - 3y_p'' + 3y_p' - y_p = (A-3A+3A-A)x^3e^x + (9A-18A+9A)x^2e^x$$

$$+ (18A-18A)xe^x + (6A)e^x - Cx + (3C-B)$$

$$= 6Ae^x - Cx + (3C-B)$$

$$= e^x - x + 16.$$

Matching coefficients: $6A = 1 \quad \Rightarrow A = 1/6$

$-C = -1 \quad C = 1$

$3C - B = 16 \quad \Rightarrow B = -13.$

So $y_p(x) = \frac{1}{6}x^3e^x - 13 + x$

and $y(x) = y_h(x) + y_p(x)$

$$= c_1e^x + c_2xe^x + c_3x^2e^x + \frac{1}{6}x^3e^x + x - 13.$$

8. Solve the IVP $y''+y=8\cos(2x)-4\sin(x)$ $\begin{cases} y(\frac{\pi}{2})=-1 \\ y'(\frac{\pi}{2})=0 \end{cases}$

First solve $y_h(x)$.

$y''+y=0$

Assume $y(x)=e^{rx}$

$y'(x)=re^{rx}$

$y''(x)=r^2e^{rx}$

$r^2+1=0$

$r=\pm i$

$y_1(x)=\cos(x)$

$y_2(x)=\sin(x)$.

$y_h(x)=c_1\cos(x)+c_2\sin(x)$.

$g(x)=8\cos(2x)-4\sin(x)$.

$8\cos(2x) \to r=\pm 2i$

$4\sin(x) \to r=\pm i$

Characteristic Eq: $(r^2+1)(r^2+4)=0$

Annihilator: $(D^2+1)(D^2+4)$.

$(D^2+1)y=8\cos(2x)-4\sin(x)$.

$\Rightarrow (D^2+1)(D^2+4)(D^2+1)y=(D^2+1)(D^2+4)[8\cos(2x)-4\sin(x)]=0$

6th order, linear, homogeneous, with constant coefficients

Assume $y(x)=e^{rx}$

$\Rightarrow (r^2+1)(r^2+4)(r^2+1)=0$

Roots: $r=\pm i$ mult 2

$r=\pm 2i$

So $y(x) = y_h + y_p$

$$= \underbrace{c_1\cos(x) + c_2\sin(x)}_{y_h} + \underbrace{c_3\cos(2x) + c_4\sin(2x) + c_5 x\cos(x) + c_6 x\sin(x)}_{y_p}.$$

Now $y_p(x) = A\cos(2x) + B\sin(2x) + Cx\cos(x) + Ex\sin(x)$.

$y_p'(x) = -2A\sin(2x) + 2B\cos(2x) + C\cos(x) - Cx\sin(x) + E\sin(x) + Ex\cos(x)$

$y_p''(x) = -4A\cos(2x) - 4B\sin(2x) - C\sin(x) - C\sin(x) - Cx\cos(x)$
$\quad + E\cos(x) + E\cos(x) - Ex\sin(x)$.
$\quad = -4A\cos(2x) - 4B\sin(2x) - 2C\sin(x) - Cx\cos(x) + 2E\cos(x) - Ex\sin(x)$.

$y_p'' + y_p = (-C + C)x\cos(x) + (-E + E)x\sin(x) + 2E\cos(x) - 2C\sin(x)$
$\quad + (-4A + A)\cos(2x) + (-4B + B)\sin(2x)$
$\quad = 2E\cos(x) - 2C\sin(x) - 3A\cos(2x) - 3B\sin(2x)$

Matching coefficients: $2E = 0 \Rightarrow E = 0$
$-2C = -4 \Rightarrow C = 2$
$-3A = 8 \Rightarrow A = -8/3$
$-3B = 0 \Rightarrow B = 0$

So $y_p(x) = -\frac{8}{3}\cos(2x) + 2x\cos(x)$.

Then $y(x) = y_h + y_p$.
$\quad = c_1\cos(x) + c_2\sin(x) - \frac{8}{3}\cos(2x) + 2x\cos(x)$

$y(\frac{\pi}{2}) = c_1(0) + c_2(1) - \frac{8}{3}(-1) + 2(\frac{\pi}{2})(0) = -1$
$\Rightarrow c_2 + \frac{8}{3} = -1$
$\Rightarrow c_2 = \frac{-11}{3}$

$y(x) = c_1\cos(x) - \frac{11}{3}\sin(x) - \frac{8}{3}\cos(2x) + 2x\cos(x)$

$y'(x) = -c_1\sin(x) - \frac{11}{3}\cos(x) + \frac{16}{3}\sin(2x) + 2\cos(x) - 2x\sin(x)$

$y'(\frac{\pi}{2}) = -c_1(1) - 0 + 0 + 0 - 2(\frac{\pi}{2})(1) = 0 \Rightarrow c_1 = -\pi$.

So $y(x) = -\pi\cos(x) - \frac{11}{3}\sin(x) - \frac{8}{3}\cos(2x) + 2x\cos(x)$