4.6 Variation of Parameters.

When can we use VOP to solve nonhomogeneous equations?
We need a linear differential equation (here's the 2nd order case) $a_2(x)y''+a_1(x)y'+a_0(x)y=g(x)$,
when we put the DE into standard form

$$y''+P(x)y'+Q(x)y=f(x)$$

$P(x)$, $Q(x)$, and $f(x)$ must be continuous on a common interval I.

This means we can solve problems from (4.5) where we have constant coefficients as well as when we have variable coefficients (4.7 - Cauchy-Euler).

Solve the following differential equations by VOP.

1. $3y''-6y'+6y=e^x\sec(x)$. $\left(\pi, \frac{3\pi}{2}\right)$

Note: This DE is not in standard form. We must do before we can solve for y_p, or else you'll end up with something that might not satisfy the right hand side of the equation (see next example).

First solve for y_h.

$3y''-6y'+6y=0$

Assume $y(x)=e^{rx}$

$$y'(x)=re^{rx}$$

$$y''(x)=r^2e^{rx}$$

$3r^2-6r+6=0$

$r^2-2r+2=0$

$$r=\frac{2\pm\sqrt{4-4(2)}}{2}=\frac{2\pm\sqrt{4}\,i}{2}=\underset{\alpha}{1}\pm\underset{\beta}{1}i.$$

$$\left.\begin{aligned} y_1(x) &= e^x\cos(x) \\ y_2(x) &= e^x\sin(x). \end{aligned}\right\}$$ FSS for the homogeneous eq.

Putting the DE in standard form, $y'' - 2y' + 2y = \frac{1}{3}e^x\sec(x)$

$$W(y_1, y_2) = \begin{vmatrix} e^x\cos(x) & e^x\sin(x) \\ e^x\cos(x) - e^x\sin(x) & e^x\sin(x) + e^x\cos(x) \end{vmatrix}$$

$$= e^x\cos(x)(e^x\sin(x) + e^x\cos(x)) - e^x\sin(x)(e^x\cos(x) - e^x\sin(x))$$

$$= e^{2x}\cos(x)\sin(x) + e^{2x}\cos^2(x) - e^{2x}\sin(x)\cos(x) + e^{2x}\sin^2(x)$$

$$= e^{2x} \neq 0$$

Assume that $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

where $u_1' = \dfrac{W_1}{W(y_1, y_2)}$ and $u_2' = \dfrac{W_2}{W(y_1, y_2)}$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^x\sin(x) \\ \frac{1}{3}e^x\sec(x) & e^x\sin(x) + e^x\cos(x) \end{vmatrix} = -\frac{1}{3}e^{2x}\tan(x)$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} e^x\cos(x) & 0 \\ e^x\cos(x) - e^x\sin(x) & \frac{1}{3}e^x\sec(x) \end{vmatrix} = \frac{1}{3}e^{2x}$$

So $u_1' = \dfrac{-\frac{1}{3}e^{2x}\tan(x)}{e^{2x}} = -\frac{1}{3}\tan(x)$

$u_2' = \dfrac{\frac{1}{3}e^{2x}}{e^{2x}} = \frac{1}{3}$

$u_1(x) = -\frac{1}{3}\int \tan(x)\,dx = -\frac{1}{3}\ln|\cos(x)|$

$u_2(x) = \frac{1}{3}\int dx = \frac{1}{3}x$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$= -\frac{1}{3}\ln|\cos(x)|e^x\cos(x) + \frac{1}{3}x e^x \sin(x)$$

So $y(x) = y_h + y_p$
$$= c_1 e^x \cos(x) + c_2 e^x \sin(x) - \frac{e^x}{3}\ln|\cos(x)|\cos(x) + \frac{x}{3}e^x \sin(x)$$

2. $2y'' + y' + y = x+1 \quad (-\infty, \infty)$ A case in what not to do.

First solve for y_h.

$2y'' + y' + y = 0$

Assume $y(x) = e^{rx}$

$2r^2 + r + 1 = 0$

$$r = \frac{-1 \pm \sqrt{1-4(1)(2)}}{2(2)} = \frac{-1 \pm 3}{4} = -1 \text{ or } \frac{1}{2}$$

$y_1(x) = e^{-x}$

$y_2(x) = e^{\frac{1}{2}x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{\frac{1}{2}x} \\ -e^{-x} & \frac{1}{2}e^{\frac{1}{2}x} \end{vmatrix} = \frac{1}{2}e^{-\frac{1}{2}x} + e^{-\frac{1}{2}x} = \frac{3}{2}e^{-\frac{1}{2}x}$$

What happens if we don't put the DE into standard form? Assume $f(x) = x+1$.

$$W_1 = \begin{vmatrix} 0 & e^{\frac{1}{2}x} \\ x+1 & \frac{1}{2}e^{\frac{1}{2}x} \end{vmatrix} = -(x+1)e^{\frac{1}{2}x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & x+1 \end{vmatrix} = (x+1)e^{-x}$$

$$u_1' = \frac{W_1}{W(y_1,y_2)} = \frac{-(x+1)e^{1/2x}}{\frac{3}{2}e^{-1/2x}} = \frac{-2}{3}(x+1)e^{x}$$

$$u_2' = \frac{W_2}{W(y_1,y_2)} = \frac{(x+1)e^{-x}}{\frac{3}{2}e^{-1/2x}} = \frac{2}{3}(x+1)e^{-\frac{1}{2}x}$$

$$u_1(x) = -\frac{2}{3}\int (x+1)e^{x}\,dx = -\frac{2}{3}\left[(x+1)e^{x} - \int e^{x}dx\right] = -\frac{2}{3}(x+1)e^{x} + \frac{2}{3}e^{x} = \frac{-2xe^{x}}{3}.$$

$u = (x+1)$ $\quad dv = e^x$

$du = dx$ $\quad v = e^x$

$$u_2(x) = \frac{2}{3}\int (x+1)e^{-1/2x}dx = \frac{2}{3}\left[-2(x+1)e^{-\frac{1}{2}x} + 2\int e^{-1/2x}dx\right] = -\frac{4}{3}(x+3)e^{-\frac{1}{2}x}$$

$u = (x+1)$ $\quad dv = e^{-1/2x}$

$du = dx$ $\quad v = -2e^{-1/2x}$

$$\begin{aligned} y_p(x) &= u_1y_1 + u_2y_2 \\ &= -\tfrac{2}{3}xe^{x}(e^{-x}) - \tfrac{4}{3}(x+3)e^{-\frac{1}{2}x}e^{\frac{1}{2}x} \\ &= \tfrac{-2}{3}x - \tfrac{4}{3}x - 4 \\ &= -2x-4. \end{aligned}$$

$y_p'(x) = -2$

$y_p''(x) = 0$

Plugging "y_p" into the DE, we have

$2y_p'' + y_p' - y_p = 0 - 2 - (-2x-4) = 2x+2 \neq x+1$

So we have ended up with something that does not satisfy the right hand side. So this cannot be a particular solution to the DE.

Instead put the DE into standard form: $y'' + \frac{1}{2}y' - \frac{1}{2}y = \frac{1}{2}(x+1)$

Assume $y_p = u_1 y_1 + u_2 y_2$.

$u_1' =$

$$W_1 = \begin{vmatrix} 0 & e^{\frac{1}{2}x} \\ \frac{1}{2}(x+1) & \frac{1}{2}e^{\frac{1}{2}x} \end{vmatrix} = -\frac{1}{2}(x+1)e^{\frac{1}{2}x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{2}(x+1) \end{vmatrix} = \frac{1}{2}(x+1)e^{-x}$$

$$\Rightarrow u_1' = -\frac{1}{3}(x+1)e^{x}$$

$$u_2' = \frac{1}{3}(x+1)e^{-\frac{1}{2}x}$$

and $u_1(x) = \frac{-xe^{x}}{3}$

$$u_2(x) = -\frac{2}{3}(x+1)e^{-\frac{1}{2}x}$$

So $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

$$= -\frac{x}{3}e^{x}(e^{-x}) - \frac{2}{3}(x+1)e^{-\frac{1}{2}x}(e^{\frac{1}{2}x})$$

$$= -x-2.$$

$y_p'(x) = -1$

$y_p''(x) = 0$

$2y_p'' + y_p' - y_p = -1-(-x-2) = x+1$. Since this satisfies the right hand side of the equation, this must be our particular solution

Now $y(x) = y_h + y_p$

$$= c_1e^{-x} + c_2e^{\frac{1}{2}x} - x - 2.$$

Note: The previous problem could have also been solved using MUC(4.5) whereas (1) can only be solved using VOP with the methods we have available.

3. $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$ Note: already in standard form.

First solve for y_h.

$y''+3y'+2y=0$

Assume $y(x) = e^{rx}$

$r^2+3r+2=0$

$(r+2)(r+1)=0$

$r=-2,-1.$

$y_1(x) = e^{-2x}$

$y_2(x) = e^{-x}$

$$W(y_1,y_2) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} = e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix} = \frac{-e^{-x}}{1+e^x}$$

$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-2x}}{1+e^x}$$

$$u_1' = \frac{W_1}{W(y_1,y_2)} = \frac{\frac{-e^{-x}}{1+e^x}}{e^{-3x}} = \frac{-e^{2x}}{1+e^x}$$

$$u_2' = \frac{W_2}{W(y_1,y_2)} = \frac{\frac{e^{-2x}}{1+e^x}}{e^{-3x}} = \frac{e^x}{1+e^x}$$

$$u_2(x) = \int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)$$

$u = 1+e^x$

$du = e^x$

$$u_1(x) = -\int \frac{e^{2x}}{1+e^x}\,dx = -\int\left(e^x - \frac{e^x}{e^x+1}\right)dx = -e^x + \ln(1+e^x).$$

$$\begin{array}{r|l} & e^x \\ \hline e^x+1 & e^{2x} \\ & \underline{e^{2x}+e^x} \\ & \quad -e^x \end{array}$$

Now $y_p = u_1 y_1 + u_2 y_2$

$= (-e^x + \ln(1+e^x))\,e^{-2x} + \ln(1+e^x)e^{-x}$

$= -e^{-x} + e^{-2x}\ln(1+e^x) + e^{-x}\ln(1+e^x)$

y_h since $y_h = c_1 e^{-2x} + c_2 e^{-x}$

$= (e^{-x} + e^{-2x})\ln(1+e^x)$

So $y(x) = y_h + y_p$

$= c_1 e^{-2x} + c_2 e^{-x} + (e^{-x} + e^{-2x})\ln(1+e^x).$