4-7 Cauchy-Euler Equations.

A Cauchy-Euler equation is an equation such that in each term, the degree of x is equal to the order of y. In the 2nd order case, we have

$$ax^2y'' + bxy' + cy = g(x) \qquad (1)$$

where a, b, c are constants

g(x) is continuous function on some interval I_0.

To solve for the homogeneous equation assume $y(x) = x^r$

So $y'(x) = rx^{r-1}$

$y''(x) = r(r-1)x^{r-2}$

Then $ax^2(r(r-1)x^{r-2}) + brxx^{r-1} + cx^r = 0$

$ar(r-1)x^r + brx^r + cx^r = 0$

$x^r[ar^2 - ar + br + c] = 0$

Note since the interval of definition is $(-\infty, 0)$ or $(0, \infty)$

We have $ar^2 + (b-a)r + c = 0$ as our characteristic equation.

Solve the following nonhomogeneous differential equations.

Note: for Cauchy-Euler problems, in general you can only solve with VOP.

1. $xy'' - 4y' = x^4$

Since the degree of x does not match the order of y, this is not Cauchy-Euler. However since x is never 0, we can multiply both sides by x to get

$x^2y'' - 4xy' = x^5$

Assume $y(x) = x^r$

$y'(x) = rx^{r-1}$

$y''(x) = r(r-1)x^{r-2}$

Solve for $x^2y'' - 4xy' = 0$

$x^2(r(r-1)x^{r-2} - 4x(rx^{r-1}) = 0$

$x^r[r(r-1) - 4r] = 0$

$r^2 - r - 4r = 0$

$r^2 - 5r = 0$

$r(r-5) = 0$

$r = 0, 5$

$y_1(x) = x^0 = 1$

$y_2(x) = x^5$

$y_h(x) = c_1 + c_2 x^5$

Putting the DE in standard form, $y'' - \frac{4}{x} y' = x^3$

Assume $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$.

$W(y_1, y_2) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix} = 0 - x^8 = -x^8$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix} = x^3 - 0 = x^3$

$u_1'(x) = \frac{W_1}{W(y_1, y_2)} = \frac{-x^8}{5x^4} = \frac{-x^4}{5}$

$u_2'(x) = \frac{W_2}{W(y_1, y_2)} = \frac{x^3}{5x^4} = \frac{1}{5x}$.

$u_1(x) = \frac{-1}{5} \int x^4 dx = -\frac{1}{5}\left(\frac{x^5}{5}\right) = \frac{-x^5}{25}$

$u_2(x) = \frac{1}{5} \int \frac{dx}{x} = \frac{1}{5} \ln x$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$= -\frac{x^5}{25}(1) + \frac{\ln(x)}{5}x^5 = \frac{x^5 \ln(x)}{5}$$

absorbed into y_2

Now $y(x) = y_h + y_p = c_1 + c_2x^5 + \frac{x^5\ln(x)}{5}$

We could have solved this equation by a change of variables. Let $w = y'$
$w' = y''$

$xw' - 4w = x^4$

Putting the DE in standard form, we have

$$\frac{dw}{dx} - \frac{4w}{x} = x^3$$

This a first order, linear, nonhomogeneous equation with variable coefficients, and nonseparable.

Here we can use the integrating factor IF: $e^{\int P(x)dx}$

$P(x) = \frac{-4}{x} \Rightarrow IF = e^{-4\int x^{-1}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$

$$x^{-4}\left(\frac{dw}{dx} - \frac{4}{x}w = x^3\right)$$

$$x^{-4}\frac{dw}{dx} - \frac{4}{x^5}w = \frac{1}{x}$$

$$\frac{d}{dx}\left[x^{-4}w\right] = \frac{1}{x}$$

$$\int \frac{d}{dx}\left[x^{-4}w\right]dx = \int \frac{dx}{x}$$

$$x^{-4}w = \ln(x) + c_1.$$

$$w = x^4 \ln(x) + c_1 x^4$$

$$\frac{dy'}{dx} = x^4 \ln(x) + c_1 x^4$$

$$\int dy = \int (x^4 \ln(x) + c_1 x^4)\,dx$$

$$s = \ln x \qquad dt = x^4$$

$$ds = x^{-1} \qquad t = \tfrac{1}{5}x^5$$

$$y(x) = \frac{x^5 \ln(x)}{5} - \int \frac{x^4}{5}dx + c_1 \int x^4 dx$$

$$= \frac{x^5 \ln(x)}{5} - \frac{x^5}{25} + c_1 \frac{x^5}{5} + c_2$$

$$= c_2 + c_3 x^5 + \frac{x^5 \ln(x)}{5}$$

2. $2x^2y'' + 5xy' + y = x^2 - x$ Cauchy-Euler.

Solve for $2x^2y'' + 5xy' + y = 0$

Assume $y(x) = x^r$

$$y'(x) = rx^{r-1}$$

$$y''(x) = r(r-1)x^{r-2}$$

$$2x^2(r(r-1)x^{r-2}) + 5x(rx^{r-1}) + x^r = 0$$

$$x^r[2r(r-1) + 5r + 1] = 0$$

$$2r^2 - 2r + 5r + 1 = 0$$

$$2r^2 + 3r + 1 = 0$$

$$r = \frac{-3 \pm \sqrt{9 - 4(2)}}{4} = \frac{-3 \pm 1}{4} = -\frac{1}{2}, -1.$$

$y_1(x) = x^{-1/2}$

$y_2(x) = x^{-1}$

$\Rightarrow y_h = c_1 y_1 + c_2 y_2 = c_1 x^{-1/2} + c_2 x^{-1}$

Put the DE in standard form: $y'' + \frac{5}{2x} y' + \frac{y}{2x^2} = \underbrace{\frac{1}{2} - \frac{1}{2x}}_{f(x)}$

$$W(y_1, y_2) = \begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix} = -x^{-5/2} - \left(-\tfrac{1}{2}x^{-5/2}\right) = \frac{-x^{-5/2}}{2}$$

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ \frac{1}{2} - \frac{1}{2x} & -x^{-2} \end{vmatrix} = 0 - \left(\frac{1}{2x} - \frac{1}{2x^2}\right) = \frac{^-(x-1)}{2x^2}$$

$$W_2 = \begin{vmatrix} x^{-1/2} & 0 \\ -\frac{1}{2}x^{-3/2} & \frac{1}{2} - \frac{1}{2x} \end{vmatrix} = x^{-1/2}\left(\frac{1}{2} - \frac{1}{2x}\right) - 0 = \frac{x-1}{2x^{3/2}}$$

$$\Rightarrow u_1'(x) = \frac{W_1}{W(y_1, y_2)} = \frac{\frac{x-1}{2x^2}}{-\frac{1}{2}x^{-5/2}} = {}^-\left(\frac{x-1}{2x^2}\right)\left(\frac{-2x^{5/2}}{1}\right) = (x-1)x^{1/2}$$

$$u_2'(x) = \frac{W_2}{W(y_1, y_2)} = \frac{\frac{x-1}{2x^{3/2}}}{-\frac{1}{2}x^{-5/2}} = \left(\frac{x-1}{2x^{3/2}}\right)\left(-2x^{5/2}\right) = -(x-1)x$$

Then $u_1(x) = \int (x^{3/2} - x^{1/2})\,dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$

$u_2(x) = \int (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}$

Now $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$$= \left(\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right)x^{-1/2} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)x^{-1}$$

$$= \frac{2}{5}x^2 - \frac{2}{3}x + \frac{x}{2} - \frac{x^2}{3}$$

$$= \frac{x^2}{15} - \frac{x}{6}$$

So $y(x) = y_h(x) + y_p(x)$

$$= c_1 x^{-1/2} + c_2 x^{-1} + \frac{x^2}{15} - \frac{x}{6}$$

3. Use the substitution $x = e^t$ to transform $x^2y'' + 9xy' - 20y = 0$ to a DE with constant coefficients.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t = x\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{dy}{dx} + x\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{dy}{dt} + x\frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{dy}{dt} + x\frac{d^2y}{dx^2}e^t$$

$$= x^2\frac{d^2y}{dx^2} + \frac{dy}{dt}$$

So $x\frac{dy}{dx} = \frac{dy}{dt}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

Substituting into the DE we have

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 9\frac{dy}{dt} - 20y = 0$$

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} - 20y = 0.$$