4-8 Solving Systems of Linear Equations by Elimination.

Solving 2x2 Systems.

$$L_1[x] + L_2[y] = f_1$$
$$L_3[x] + L_4[y] = f_2$$

where L_i is the differential operator on x, y.

1. Put system in differential operator notation
2. Eliminate one variable, say $y(t)$, and solve for $x(t)$
3. (<u>If Possible</u>) Derive an equation for $y(t)$ in terms of $x(t)$ where no derivatives of $y(t)$ are present. Solve for $y(t)$.

Otherwise

4. Eliminate $x(t)$ and solve $y(t)$.
5. Plug $x(t)$ and $y(t)$ into one the equations in the system to get $x(t)$ and $y(t)$ in terms of the same parameters.

I. Solve the given system of DEs by elimination.

1. $\frac{dx}{dt} = -y + t$

$\frac{dy}{dt} = x - t$

$$\Rightarrow \begin{aligned} Dx + y &= t \\ -x + Dy &= -t \end{aligned}$$

Apply D to 2nd equation and add equations together.

$$\begin{aligned} Dx + y &= t \\ -Dx + D^2y &= -1 \\ \hline D^2y + y &= t - 1 \end{aligned}$$

$(D^2+1)y = t-1$ or $y''+y = t-1$

Solve for $y_h(t)$ Assume $y(t)=e^{rt}$

$y''+y=0$

$r^2+1=0$

$r=\pm i$

$y_1(t)=\cos(t)$

$y_2(t)=\sin(t)$.

Now since each term on the right corresponds to a root from the characteristic equation $ar^2+br+c=0$, we have 2 possible methods to solve for y_p:

1. MUC - need an annihilator for RHS.
2. VOP - need to integrate by parts.

<u>Method 1</u>: MUC.

$t-1 \Rightarrow r=0$ mult 2.

$D^2(t-1)=0$

$D^2(D^2+1)y = D^2(t-1)=0$

Assume $y(t)=e^{rt}$

$r^2(r^2+1)=0$

$$y(t) = \underbrace{c_1\cos(t)+c_2\sin(t)}_{y_h} + \underbrace{c_3+c_4 t}_{y_p}$$

$y_p(t) = A+Bt$

$y_p'(t)=B$

$y_p''(t)=0$

So $y_p''+y_p = 0+A+Bt = t-1$

$A=-1$

$B=1$

So $y_p = t-1$.

<u>Method 2</u>: VOP.

Note: DE is already in standard form.

$$W(y_1, y_2) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1.$$

$$W_1 = \begin{vmatrix} 0 & \sin(t) \\ t-1 & \cos(t) \end{vmatrix} = -(t-1)\sin(t)$$

$$W_2 = \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & t-1 \end{vmatrix} = (t-1)\cos(t).$$

So $u_1'(t) = -(t-1)\sin(t)$

$u_2'(t) = (t-1)\cos(t)$

$$u_1(t) = \int (1-t)\sin(t)dt = (t-1)\cos(t) - \int \cos(t)dt = (t-1)\cos(t) - \sin(t)$$

$u = (1-t)$ $\quad dv = \sin(t)$

$du = -dt$ $\quad v = -\cos(t)$

$$u_2(t) = \int (t-1)\cos(t)dt = (t-1)\sin(t) - \int \sin(t)dt = (t-1)\sin(t) + \cos(t)$$

$u = (t-1)$ $\quad dv = \cos(t)$

$du = dt$ $\quad v = \sin(t)$

Now $y_p(t) = u_1 y_1 + u_2 y_2$

$$= [(t-1)\cos(t) - \sin(t)]\cos(t) + [(t-1)\sin(t) + \cos(t)]\sin(t)$$

$$= (t-1)\cos^2(t) - \cancel{\sin(t)\cos(t)} + (t-1)\sin^2(t) + \cancel{\cos(t)\sin(t)}$$

$$= (t-1)$$

Now $y(t) = y_h + y_p$

$= c_1\cos(t) + c_2\sin(t) + t - 1.$

Note: $x = Dy + t$

$y'(t) = -c_1\sin(t) + c_2\cos(t) + 1.$

So $x(t) = -c_1\sin(t) + c_2\cos(t) + t + 1.$

Otherwise: Eliminate $y(t)$ and solve for $x(t)$. Then plug $x(t)$ and $y(t)$ to get $x(t)$ and $y(t)$ in terms of same parameters.

So the solution to the system is

$$\begin{cases} x(t) = -c_1\sin(t) + c_2\cos(t) + t + 1 \\ y(t) = c_1\cos(t) + c_2\sin(t) + t - 1. \end{cases}$$

2. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$

$\Rightarrow (D^2+5)x + Dy = 0$

$(D+1)x + (D-4)y = 0$

Apply $(D+1)$ to the first equation and $-(D^2+5)$ to the second to eliminate $x(t)$.

$$(D+1)[(D^2+5)x + Dy = 0]$$
$$\underline{-(D^2+5)[(D+1)x + (D-4)y = 0]}$$
$$D(D+1)y - (D^2+5)(D-4)y = 0$$

Now $\left(D(D+1) - (D^2+5)(D-4)\right)y = 0$

$$(D^2+D - (D^3-4D^2+5D-20))\,y = 0$$

$$(-D^3+5D^2-4D+20)y = 0$$

$$(D^3-5D^2+4D-20)y = 0$$

$$[D^2(D-5)+4(D-5)]\,y = 0$$

$$(D^2+4)(D-5)\,y = 0$$

Assume $y(t) = e^{rt}$

$y(t) = c_1e^{5t} + c_2\cos(2t) + c_3\sin(2t)$.

Apply $(D-4)$ to the first equation and $-D$ to the second to eliminate $y(t)$.

$(D-4)[(D^2+5)x + Dy = 0]$

$-D[(D+1)x + (D-4)y = 0]$.

$(D-4)(D^2+5)x - D(D+1)x = 0$

$[(D-4)(D^2+5) - D(D+1)]x = 0$

$[D^3+5D-4D^2-20-D^2-D]x = 0$

$(D^3-5D^2+4D-20)x = 0$

$(D^2+4)(D-5)x = 0$

Assume $x(t) = e^{rt}$

$x(t) = b_1e^{5t} + b_2\cos(2t) + b_3\sin(2t)$

Now need to $x(t)$ and $y(t)$ in terms of the same parameters.

Plug into the second equation : $(D+1)x + (D-4)y = 0$

$x(t) = b_1e^{5t} + b_2\cos(2t) + b_3\sin(2t)$

$x'(t) = 5b_1e^{5t} - 2b_2\sin(2t) + 2b_3\cos(2t)$

$y(t) = c_1e^{5t} + c_2\cos(2t) + c_3\sin(2t)$

$y'(t) = 5c_1e^{5t} - 2c_2\sin(2t) + 2c_3\cos(2t)$

$x' + x + y' - 4y = 0$

$(5b_1 + b_1)e^{5t} + (b_2 + 2b_3)\cos(2t) + (b_3 - 2b_2)\sin(2t) + (5c_1 + c_1)e^{5t}$

$+ (c_2 + 2c_3)\cos(2t) + (c_3 - 2c_2)\sin(2t) = 0$

$6b_1 e^{5t} + (b_2 + 2b_3)\cos(2t) + (b_3 - 2b_2)\sin(2t) = -6c_1 e^{5t} - (c_2 + 2c_3)\cos(2t) - (c_3 - 2c_2)\sin(2t)$

$6b_1 = -6c_1 \quad \Rightarrow b_1 = -c_1$

$\left.\begin{aligned} b_2 + 2b_3 &= -c_2 - 2c_3 \\ -2b_2 + b_3 &= 2c_2 - c_3 \end{aligned}\right\}$ 2 eq with 2 unknowns.

$2(b_2 + 2b_3 = -c_2 - 2c_3)$

$\underline{-2b_2 + b_3 = 2c_2 - c_3}$

$5b_3 = -5c_3 \quad \Rightarrow b_3 = -c_3.$

$\Rightarrow b_2 = -c_2.$

So the solution to the system is

$$\begin{cases} x(t) = -c_1 e^{5t} - c_2\cos(t) - c_3\sin(t) \\ y(t) = c_1 e^{5t} + c_2\cos(t) + c_3\sin(t). \end{cases}$$