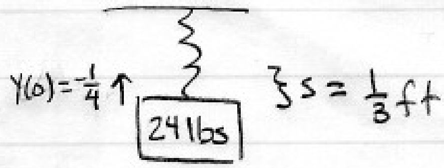
5-1 Linear Models: IVPs.

1. Free undamped motion: A mass weighing 24 lbs, attached to the end of a spring, stretches it 4 in. Initially, the mass is released from rest from a point 3 in above the equilibrium position. Find the equation of motion.




$$m = \frac{W}{g} = \frac{24}{32} = \frac{3}{4}.$$

$$s = 4\text{in}\left(\frac{1\text{ft}}{12\text{in}}\right) = \frac{1}{3}\text{ft}.$$

$$Ks = mg \implies \frac{1}{3}K = 24 \implies K = 72.$$

So the IVP is $\frac{3}{4}y'' + 72y = 0$

$$\begin{cases} y(0) = \frac{-3}{12} = \frac{-1}{4} & \text{initial displacement.} \\ y'(0) = 0 & \text{initial velocity.} \end{cases}$$

$y'' + 96y = 0$

Assume $y(t) = e^{rt}$

$y'(t) = re^{rt}$

$y''(t) = r^2 e^{rt}$

$\implies r^2 + 96 = 0$

$r^2 = -96$

$r = \pm\sqrt{96}\,i = \pm 4\sqrt{6}\,i$

$4\sqrt{6}$ is the natural frequency, $\omega$

$y(t) = c_1\cos(4\sqrt{6}\,t) + c_2\sin(4\sqrt{6}\,t)$

$y(0) = c_1 + 0 = -\frac{1}{4} \Rightarrow c_1 = -\frac{1}{4}$

So $y(t) = -\frac{1}{4}\cos(4\sqrt{6}t) + c_2 \sin(4\sqrt{6}t)$

$y'(t) = \sqrt{6}\sin(4\sqrt{6}t) + 4\sqrt{6}c_2\cos(4\sqrt{6}t)$

$y'(0) = 0 + 4\sqrt{6}c_2 = 0 \Rightarrow c_2 = 0$

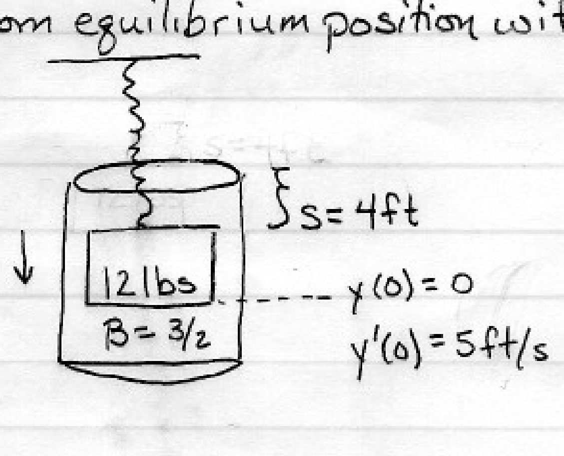
Now $y(t) = -\frac{1}{4}\cos(4\sqrt{6}t)$

Amplitude of motion: $A = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(-\frac{1}{4}\right)^2 + 0} = \frac{1}{4}$

Period of motion: $T = \frac{2\pi}{\omega} = \frac{2\pi}{4\sqrt{6}} = \frac{\pi}{2\sqrt{6}}$

Frequency: $f = \frac{1}{T} = \frac{2\sqrt{6}}{\pi}$.

2. Free damped motion: A 4ft spring measures 8ft long after an object weighing 12 lbs is attached to it. The medium through which the object moves offers a damping force numerically equal to 3/2 times the instantaneous velocity. Find the position $y(t)$ for any time $t$ if the object is initially released from equilibrium position with a downward velocity of 5 ft/s.



$m = \frac{W}{g} = \frac{12\,lbs}{32\,ft/s} = \frac{3}{8}$ slug

$Ks = mg = W \Rightarrow 4K = 12$

$\Rightarrow K = 3$

So $my'' + \beta y' + Ky = 0 \Rightarrow \frac{3}{8}y'' + \frac{3}{2}y' + 3y = 0$

Then the IVP is $3y'' + 12y' + 24y = 0$ $\begin{cases} y(0) = 0 & \text{initial displacement} \\ y'(0) = 5 & \text{initial velocity.} \end{cases}$

Assume $y(t) = e^{rt}$

$y'(t) = re^{rt}$

$y''(t) = r^2 e^{rt}$

$\Rightarrow 3r^2 + 12r + 24 = 0$

$r^2 + 4r + 8 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4(1)(8)}}{2(1)} = \frac{-4 \pm 4i}{2} = \underset{\alpha}{-2} \pm \underset{\beta}{2}i$

Now $y_1(t) = e^{-2t}\cos(2t)$

$y_2(t) = e^{-2t}\sin(2t)$

$y(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)$

$y(0) = c_1(1) + c_2(0) = 0 \quad \Rightarrow c_1 = 0$

So $y(t) = c_2 e^{-2t}\sin(2t)$

$y'(t) = -2c_2 e^{-2t}\sin(2t) + 2c_2 e^{-2t}\cos(2t)$

$y'(0) = 0 + 2c_2 = 5 \quad \Rightarrow c_2 = \frac{5}{2}$

Now $y(t) = \frac{5}{2} e^{-2t}\sin(2t)$

Damped amplitude: $Ae^{-\lambda t} = \sqrt{c_1^2+c_2^2}\, e^{-\alpha t}$

$$= \sqrt{0+\left(\frac{5}{2}\right)^2}\, e^{-2t}$$

$$= \frac{5}{2} e^{-2t}$$

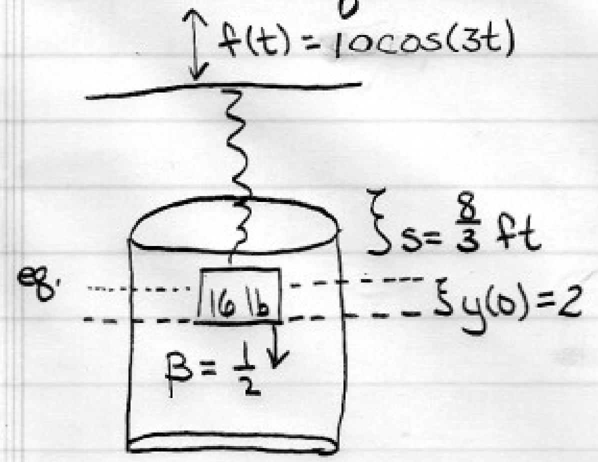
Quasi Period: $\dfrac{2\pi}{\sqrt{\omega^2-\lambda^2}} = \dfrac{2\pi}{\beta}$

$$= \frac{2\pi}{2}$$

$$= \pi$$

Quasi frequency: $\dfrac{\sqrt{\omega^2-\lambda^2}}{2\pi} = \dfrac{1}{\pi}.$

3. <u>Driven motion</u>: An object weighing 16 lbs stretches a spring $\frac{8}{3}$ ft. The object is initially released from rest at a point 2 ft below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to $\frac{1}{2}$ times the instantaneous velocity. Find the position $y(t)$ at any time $t$ if the object is driven by an external force equal to $f(t) = 10\cos(3t)$



$m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$

$Ks = mg = W \Rightarrow \frac{8}{3}K = 16$

$\Rightarrow K = 6$

$\beta = \frac{1}{2}$

Then $my'' + \beta y' + Ky = f(t) \Rightarrow \frac{1}{2}y'' + \frac{1}{2}y' + 6y = 10\cos(3t)$

So the IVP is $y'' + y' + 12y = 20\cos(3t)$

$\begin{cases} y(0) = 2 & \text{initial displacement} \\ y'(0) = 0 & \text{initial velocity} \end{cases}$

First, solve $y'' + y' + 12y = 0$

Assume $y(t) = e^{rt}$

$y'(t) = re^{rt}$

$y''(t) = r^2 e^{rt}$

$\Rightarrow r^2 + r + 12 = 0$

$r = \frac{-1 \pm \sqrt{1 - 4(1)(12)}}{2(1)} = \frac{-1 \pm \sqrt{-47}}{2} = \frac{-1}{2} \pm \frac{\sqrt{47}}{2}i$

Now $y_1(t) = e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{47}}{2}t\right)$

$y_2(t) = e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{47}}{2}t\right)$

So $y_h(t) = e^{-\frac{1}{2}t}\left[c_1\cos\left(\frac{\sqrt{47}}{2}t\right) + c_2\sin\left(\frac{\sqrt{47}}{2}t\right)\right]$ transient solution.

Note: $\lim_{t \to \infty} y_h(t) = 0$.

Rewriting the DE in differential operator notation, we have
$(D^2+D+12)y = 20\cos(3t)$
Note: the annihilator for $\cos(3t)$ is $(D^2+9)$

Then $(D^2+9)(D^2+D+12)y = (D^2+9)[20\cos(3t)] = 0$
Assume $y(t)=e^{rt}$
$\Rightarrow (r^2+9)(r^2+r+12)=0$
$\Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{47}}{2}i, \pm 3i$

So $y(t) = \underbrace{e^{-\frac{1}{2}t}\left[c_1\cos\left(\frac{\sqrt{47}}{2}t\right)+c_2\sin\left(\frac{\sqrt{47}}{2}t\right)\right]}_{y_h} + \underbrace{c_3\cos(3t)+c_4\sin(3t)}_{y_p}$

$\Rightarrow y_p(t) = A\cos(3t)+B\sin(3t)$
$y_p'(t) = -3A\sin(3t)+3B\cos(3t)$
$y_p''(t) = -9A\cos(3t)-9B\sin(3t)$

Now plugging $y_p$ back into the original DE, we have

$$\begin{aligned} y_p''+y_p'+12y_p &= (-9A+3B+12A)\cos(3t)+(-9B-3A+12B)\sin(3t) \\ &= (3A+3B)\cos(3t)+(-3A+3B)\sin(3t) \\ &= 20\cos(3t)+0\sin(3t) \end{aligned}$$

comparing coefficients, $3A+3B=20$
$-3A+3B=0 \quad \Rightarrow A=B$

$6A=20 \Rightarrow A=\frac{10}{3}=B$

Now $y_p(t) = \frac{10}{3}\cos(3t)+\frac{10}{3}\sin(3t)$ steady-state solution.

So $y(t) = y_h + y_p$

$$= c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{47}t}{2}\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{47}t}{2}\right) + \frac{10}{3}\cos(3t) + \frac{10}{3}\sin(3t)$$

$$y(0) = c_1 + 0 + \frac{10}{3} + 0 = 2 \quad \Rightarrow c_1 = \frac{-4}{3}$$

$$y(t) = \frac{-4}{3} e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{47}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}\cos(3t) + \frac{10}{3}\sin(3t)$$

$$y'(t) = \frac{2}{3} e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{47}}{2}t\right) + \frac{2\sqrt{47}}{3} e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{47}}{2}t\right) - \frac{1}{2}c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{47}}{2}t\right)$$
$$+ \frac{\sqrt{47}}{2}c_2 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{47}}{2}t\right) - 10\sin(3t) + 10\cos(3t).$$

$$y'(0) = \frac{2}{3} + 0 - 0 + \frac{\sqrt{47}}{2}c_2 - 0 + 10 = 0$$

$$\frac{\sqrt{47}}{2}c_2 = -\frac{32}{3}$$

$$c_2 = \frac{-64}{3\sqrt{47}}$$

So $y(t) = \frac{-4}{3} e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{47}}{2}t\right) - \frac{64}{3\sqrt{47}} e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}\cos(3t) + \frac{10}{3}\cos(3t).$

4. <u>Resonance</u>: Consider the spring mass system described in (1), only now be driven by an external forcing function $f(t) = 3\cos(\gamma t)$, where $w$ is the driving frequency. Find the value $\gamma$ such that resonance occurs, as well as the position $y(t)$ at any time $t$.

From before, $my'' + ky = f(t) \Rightarrow \frac{3}{4}y'' + 72y = 3\cos(\gamma t)$.

So the IVP is
$$y'' + 96y = 4\cos(\gamma t)$$
$$\begin{cases} y(0) = -\frac{1}{4} & \text{initial displacement} \\ y'(0) = 0 & \text{initial velocity} \end{cases}$$

First, solving for $y'' + 96y = 0$:

Assume $y(t) = e^{rt}$

$y'(t) = re^{rt}$

$y''(t) = r^2 e^{rt}$

$\Rightarrow r^2 + 96 = 0$

$\Rightarrow r = \pm 4\sqrt{6}\,i$

So $4\sqrt{6}$ is the natural frequency, $w$.

<u>Note</u>: Resonance occurs when the natural frequency and the driving frequency are the same ($\gamma = w = 4\sqrt{6}$). (in other words, you can think of resonance problems like repeated root problems as in 4.5).

Now $y_h(t) = c_1\cos(4\sqrt{6}t) + c_2\sin(4\sqrt{6}t)$

Rewriting the DE in differential operator notation, we have $(D^2+96)y = 4\cos(4\sqrt{6}t)$.

Since $\gamma = \omega$, the annihilator of $4\cos(4\sqrt{6}t)$ is $D^2+96$.

Now $(D^2+96)(D^2+96)y = (D^2+96)[4\cos(4\sqrt{6}t)] = 0$

Assume $y(t) = e^{rt}$

$\Rightarrow (r^2+96)^2 = 0$

$\Rightarrow r = \pm 4\sqrt{6}i, \pm 4\sqrt{6}i$

So $y(t) = \underbrace{c_1\cos(4\sqrt{6}t) + c_2\sin(4\sqrt{6}t)}_{y_h} + \underbrace{c_3 t\cos(4\sqrt{6}t) + c_4 t\sin(4\sqrt{6}t)}_{y_p}$

$\Rightarrow y_p = At\cos(4\sqrt{6}t) + Bt\sin(4\sqrt{6}t)$

$y_p' = A\cos(4\sqrt{6}t) - 4\sqrt{6}At\sin(4\sqrt{6}t) + B\sin(4\sqrt{6}t) + 4\sqrt{6}Bt\cos(4\sqrt{6}t)$

$y_p'' = -8\sqrt{6}A\sin(4\sqrt{6}t) - 96At\cos(4\sqrt{6}t) + 8\sqrt{6}B\cos(4\sqrt{6}t) - 96Bt\sin(4\sqrt{6}t)$

Now plugging $y_p$ into the DE, we have

$$\begin{aligned} y_p'' + 96y_p &= (-96A+96A)t\cos(4\sqrt{6}t) + (-96B+96B)t\sin(4\sqrt{6}t) \\ &\quad - 8\sqrt{6}A\sin(4\sqrt{6}t) + 8\sqrt{6}B\cos(4\sqrt{6}t) \\ &= -8\sqrt{6}A\sin(4\sqrt{6}t) + 8\sqrt{6}B\cos(4\sqrt{6}t) \\ &= 4\cos(4\sqrt{6}t) + 0\sin(4\sqrt{6}t) \end{aligned}$$

$-8\sqrt{6}A = 0 \quad \Rightarrow A = 0$

$8\sqrt{6}B = 4 \quad \Rightarrow B = \dfrac{1}{2\sqrt{6}}$

So $y_p(t) = \dfrac{1}{2\sqrt{6}} t\sin(4\sqrt{6}t)$

$$y(t) = y_h + y_p$$
$$= c_1 \cos(4\sqrt{6}t) + c_2 \sin(4\sqrt{6}t) + \frac{1}{2\sqrt{6}} t \sin(4\sqrt{6}t)$$

$$y(0) = c_1(1) + c_2(0) + 0 = -\frac{1}{4} \Rightarrow c_1 = -\frac{1}{4}$$

So $y(t) = -\frac{1}{4}\cos(4\sqrt{6}t) + c_2 \sin(4\sqrt{6}t) + \frac{1}{2\sqrt{6}} t \sin(4\sqrt{6}t)$

$$y'(t) = \sqrt{6}\sin(4\sqrt{6}t) + 4\sqrt{6}c_2 \cos(4\sqrt{6}t) + \frac{1}{2\sqrt{6}} \sin(4\sqrt{6}t) + 2t\cos(4\sqrt{6}t)$$

$$y'(0) = 0 + 4\sqrt{6}c_2 + 0 + 0 = 0 \Rightarrow c_2 = 0$$

So $y(t) = -\frac{1}{4}\cos(4\sqrt{6}t) + \frac{1}{2\sqrt{6}} t \sin(4\sqrt{6}t)$