5-2 Linear Models: BVPs.

1. A beam of length L is embedded at its left end and simply supported at its right end. Suppose the beam has a constant load w_0 that is uniformly distributed along its length, ie $w(x) = w_0$, $0<x<L$. Find the deflection $y(x)$ of the beam that satisfies the equation $EI\frac{d^4y}{dx^4} = w_0$

$$\frac{d^4y}{dx^4} = \frac{w_0}{EI} \quad \text{subject to} \begin{cases} y(0)=0 \\ y'(0)=0 \end{cases} \quad \begin{matrix} y(L)=0 \\ y''(L)=0 \end{matrix}$$

Solve for $y^{(4)} = 0$.

Assume $y(x) = e^{rx}$

$\Rightarrow r^4 = 0$

$r=0$ multiplicity 4

$y_h(x) = c_1 + c_2x + c_3x^2 + c_4x^3$

Rewriting the equation in differential operator notation, we have $D^4y = \frac{w_0}{EI}$.

Find the annihilator of $\frac{w_0}{EI}$: D

$$D(D^4y) = D\left[\frac{w_0}{EI}\right] = 0$$

Assume $y(x) = e^{rx}$

$\Rightarrow r^5 = 0$

$r=0$ multiplicity 5

$$y(x) = \underbrace{c_1 + c_2x + c_3x^2 + c_4x^3}_{y_h} + \underbrace{c_5x^4}_{y_p}$$

$y_p(x) = Ax^4$
$y_p'(x) = 4Ax^3$
$y_p''(x) = 12Ax^2$
$y_p'''(x) = 24Ax$
$y_p^{(4)}(x) = 24A$

$$y_p^{(4)}(x) = \frac{w_0}{EI} \Rightarrow 24A = \frac{w_0}{EI}$$
$$\Rightarrow A = \frac{w_0}{24EI}.$$

So $y(x) = y_h(x) + y_p(x)$
$$= c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{w_0}{24EI} x^4$$

$$y'(x) = c_2 + 2c_3 x + 3c_4 x^2 + \frac{w_0}{6EI} x^3$$

$$y''(x) = 2c_3 + 6c_4 x + \frac{w_0}{2EI} x^2$$

$y(0) = c_1 + 0 + 0 + 0 + 0 = 0 \Rightarrow c_1 = 0$
$y'(0) = c_2 + 0 + 0 + 0 = 0 \Rightarrow c_2 = 0$
$$y(L) = c_3 L^2 + c_4 L^3 + \frac{w_0}{24EI} L^4 = 0 \qquad (1)$$

$$y''(L) = 2c_3 + 6c_4 L + \frac{w_0}{2EI} L^2 = 0$$
$$\Rightarrow c_3 = -\frac{1}{2}\left(6c_4 L + \frac{w_0 L^2}{2EI}\right) = -3c_4 L - \frac{w_0 L^2}{4EI} \qquad (2)$$

Substituting (2) into (1),

$$\left(-3c_4 L - \frac{w_0 L^2}{4EI}\right) L^2 + 6c_4 L^3 + \frac{w_0}{24EI} L^4 = 0$$

$$-3c_4 L^3 - \frac{w_0 L^4}{4EI} + 6c_4 L^3 + \frac{w_0 L^4}{24EI} = 0$$

$$-2c_4 L^3 - \frac{5w_0 L^4}{24EI} = 0$$

$$c_4 = -\frac{5w_0 L}{48EI}$$

$$c_3 = -3\left(\frac{-5w_0 L}{48EI}\right) L - \frac{w_0 L^2}{4EI} -$$

$$= \frac{15w_0 L^2}{48EI} - \frac{12w_0 L^2}{48EI}$$

$$= \frac{3w_0 L^2}{48EI}$$

So $y(x) = \dfrac{3w_0 L^2 x^2}{48EI} - \dfrac{5w_0 L x^3}{48EI} + \dfrac{w_0 x^4}{24EI}$

$$= \frac{w_0}{48EI} x^2 (3L - 5x + 2x^2)$$

2. Find the eigenvalues and eigenfunctions of the BVP

$y''+\lambda y=0 \quad \begin{cases} y'(0)=0 \\ y'(\pi)=0 \end{cases}$

In general, assume $y(x)=e^{rx}$

$\Rightarrow r^2+\lambda r=0$

$r=\dfrac{-0\pm\sqrt{0-4\lambda(1)}}{2(1)}$ Recall that the discriminant determines the form of $y(x)$.

Case 1: $\lambda=0 \Rightarrow r=0$ mult 2.

$y_1(x)=e^{0x}=1.$

$y_2(x)=xe^{0x}=x$

$y(x)=c_1+c_2x.$

$y'(x)=c_2$

$y'(0)=c_2=0$

$y'(\pi)=c_2=0$

So $\lambda=0$ is an eigenvalue and $y(x)=c_1$ is the corresponding eigenfunction.

Case 2: $\lambda<0$. Let $\lambda=-\alpha^2$ where $\alpha=0$

So $r=\pm\alpha$

$y_1(x)=e^{\alpha x}$

$y_2(x)=e^{-\alpha x}$

$y(x)=c_1e^{\alpha x}+c_2e^{-\alpha x}$

$y'(x)=\alpha c_1e^{\alpha x}-\alpha c_2e^{-\alpha x}$

$y'(0)=\alpha c_1-\alpha c_2=0 \quad \Rightarrow c_1=c_2$

$y'(\pi)=\alpha c_1e^{\alpha\pi}-\alpha c_2e^{-\alpha\pi}=0$

$\Rightarrow c_1e^{\alpha\pi}-c_2e^{-\alpha\pi}=0$

$\Rightarrow c_2(e^{\alpha\pi}-e^{-\alpha\pi})=0$

$\Rightarrow c_2=0=c_1$

So only the trivial solution satisfies the BCs.

Case 3: $\lambda > 0$. Let $\lambda = \alpha^2$ where $\alpha > 0$

So $r = \pm \alpha i$

$y_1(x) = \cos(\alpha x)$

$y_2(x) = \sin(\alpha x)$

So $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$y'(x) = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x)$

$y'(0) = 0 + \alpha c_2 = 0 \quad \Rightarrow \quad c_2 = 0$

So $y'(x) = -\alpha c_1 \sin(\alpha x)$

$y'(\pi) = -\alpha c_1 \sin(\alpha \pi)$

What values α is $\sin(\alpha\pi) = 0$?

$\alpha_n = 1, 2, 3, \ldots n$

Since $\lambda_n = \alpha_n^2$, $\lambda_n = n^2$ are the eigenvalues and

$y_n(x) = c_1 \cos(nx)$ are the corresponding eigenfunctions.

3. Find the eigenvalues and eigenfunctions of $y'' + 2y' + (\lambda+1)y = 0$

subject to $\begin{cases} y(0) = 0 \\ y(5) = 0 \end{cases}$

Assume $y(x) = e^{rx}$

$\Rightarrow r^2 + 2r + (\lambda + 1) = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(\lambda+1)}}{2(1)} = -1 \pm \sqrt{-\lambda}$$

Case 1: $\lambda = 0 \Rightarrow r = -1$ mult 2.

$y_1(x) = e^{-x}$

$y_2(x) = xe^{-x}$

$y(x) = c_1 e^{-x} + c_2 x e^{-x}$

$y(0) = c_1 + 0 = 0 \quad \Rightarrow c_1 = 0$

$y(x) = c_2 x e^{-x}$

$y(5) = 5c_2 e^{-5} = 0 \quad \Rightarrow c_2 = 0$

Only the trivial solution satisfies the BCs for $\lambda = 0$.

Case: $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha > 0 \Rightarrow r = -1 \pm \alpha$

$y_1(x) = e^{(-1+\alpha)x}$

$y_2(x) = e^{(-1-\alpha)x}$

$y(x) = c_1 e^{(-1+\alpha)x} + c_2 e^{(-1-\alpha)x}$

$y(0) = c_1 + c_2 = 0 \quad \Rightarrow c_1 = -c_2$

$y(5) = c_1 e^{(-1+\alpha)5} + c_2 e^{(-1-\alpha)5} = 0$

$-c_2 e^{(-1+\alpha)5} + c_2 e^{(-1-\alpha)5} = 0$

$c_2\left(-e^{(-1+\alpha)5} + e^{(-1-\alpha)5}\right) = 0$

$\Rightarrow c_2 = 0 = c_1$

Only the trivial solution satisfies the BCs for $\lambda < 0$

Case 3: $\lambda > 0$. Let $\lambda = \alpha^2$ where $\alpha > 0$. $\Rightarrow r = -1 \pm \alpha i$

$y_1(x) = e^{-x}\cos(\alpha x)$

$y_2(x) = e^{-x}\sin(\alpha x)$

$y(x) = c_1 e^{-x}\cos(\alpha x) + c_2 e^{-x}\sin(\alpha x)$

$y(0) = c_1 + 0 = 0 \quad \Rightarrow c_1 = 0$

$y(x) = c_2 e^{-x}\sin(\alpha x)$

$y(5) = c_2 e^{-5}\sin(5\alpha)$ What values of α is $\sin(5\alpha) = 0$?

$\alpha_n = \frac{n\pi}{5}$, $n = 1, 2, \ldots \Rightarrow \lambda_n = \alpha_n^2 = \left(\frac{n\pi}{5}\right)^2$, $n = 1, 2, \ldots$ are

the eigenvalues and $y_n(x) = e^{-x}\sin\left(\frac{n\pi x}{5}\right)$, $n = 1, 2, \ldots$ are

the corresponding eigenfunctions