7-1 Definition of the Laplace Transform

Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$.

1. $f(t) = \begin{cases} (2t+1), & 0 \le t \le 1. \\ 1 & t \ge 1. \end{cases}$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^1 e^{-st}(2t+1)\,dt + \int_1^\infty e^{-st}\,dt.$

$u = 2t+1 \quad dv = e^{-st}$

$du = 2 \quad v = -\frac{1}{s}e^{-st}$

$= -\frac{(2t+1)e^{-st}}{s}\Big|_0^1 + \frac{2}{s}\int_0^1 e^{-st}\,dt + \int_1^\infty e^{-st}\,dt$

$= -\frac{3e^{-s}}{s} + \frac{1}{s} + \frac{2}{s}\left[-\frac{e^{-st}}{s}\right]_0^1 - \frac{e^{-st}}{s}\Big|_1^\infty$

$= -\frac{3e^{-s}}{s} + \frac{1}{s} + \frac{2}{s}\left[-\frac{e^{-s}}{s} + \frac{1}{s}\right] - 0 + \frac{1}{s}$

$= -\frac{3e^{-s}}{s} + \frac{2}{s} - \frac{2e^{-s}}{s^2} + \frac{2}{s^2}$

2. $f(t) = e^{-2t-5}$

$\mathcal{L}\{e^{-2t-5}\} = \int_0^\infty e^{-st}e^{-2t}e^{-5}\,dt$

$= e^{-5}\int_0^\infty e^{-(s+2)t}\,dt$

$= e^{-5}\frac{e^{-(s+2)t}}{-(s+2)}\Big|_0^\infty$

$= 0 - -\frac{e^{-5}}{s+2}$

$= \frac{e^{-5}}{s+2}$ provided $s+2>0$

$s > -2$

3. $f(t) = e^t \cos(t)$

$$\mathcal{L}\{e^t \cos(t)\} = \int_0^\infty e^{-st} e^t \cos(t)\,dt$$
$$= \int_0^\infty e^{-(s-1)t} \cos(t)\,dt.$$

$u = \cos(t)$ $\quad dv = e^{-(s-1)t}$

$du = -\sin(t)$ $\quad v = -\dfrac{e^{-(s-1)t}}{s-1}$

$$= -\frac{\cos(t) e^{-(s-1)t}}{s-1}\Big|_0^\infty - \frac{1}{s-1}\int_0^\infty e^{-(s-1)t} \sin(t)\,dt$$

$$= 0 + \frac{\cos(0) e^0}{s-1} - \frac{1}{s-1}\int_0^\infty e^{-(s-1)t} \sin(t)\,dt$$

$u = \sin(t)$ $\quad dv = e^{-(s-1)t}$

$du = \cos(t)$ $\quad v = -\dfrac{e^{-(s-1)t}}{s-1}$

$$= \frac{1}{s-1} - \frac{1}{s-1}\left(-\frac{\sin(t) e^{-(s-1)t}}{s-1}\Big|_0^\infty + \frac{1}{s-1}\int_0^\infty e^{-(s-1)t} \cos(t)\,dt\right)$$

$$= \frac{1}{s-1} - \frac{1}{s-1}\left(0 + 0 + \frac{1}{s-1}\int_0^\infty e^{-(s-1)t} \cos(t)\,dt\right).$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2}\mathcal{L}\{e^t \cos(t)\}$$

So $\mathcal{L}\{e^t \cos(t)\} + \dfrac{1}{(s-1)^2}\mathcal{L}\{e^t \cos(t)\} = \dfrac{1}{s-1}$

$$\left(\frac{(s-1)^2 + 1}{(s-1)^2}\right)\mathcal{L}\{e^t \cos(t)\} = \frac{1}{s-1}$$

So $\mathcal{L}\{e^t \cos(t) = \left(\dfrac{1}{s-1}\right)\left(\dfrac{(s-1)^2}{(s-1)^2+1}\right) = \dfrac{s-1}{(s-1)^2+1}$