7.2: Inverse Transforms and Transforms of Derivatives.

Find the inverse Laplace transform of the following:

1. $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{2!}{2!}\right)\left(\frac{1}{s^3}\right)\right\}$

$= \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}$

$= \frac{1}{2}t^2$

2. $\mathcal{L}^{-1}\left\{\left(\frac{2}{s} - \frac{1}{s^3}\right)^2\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2} - \frac{4}{s^4} + \frac{1}{s^6}\right\}$.

$= 4\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\}$

$= 4\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 4\mathcal{L}^{-1}\left\{\left(\frac{3!}{3!}\right)\left(\frac{1}{s^{3+1}}\right)\right\} + \mathcal{L}^{-1}\left\{\left(\frac{5!}{5!}\right)\left(\frac{1}{s^{5+1}}\right)\right\}$

$= 4t - \frac{4}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\} + \frac{1}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^{5+1}}\right\}$

$= 4t - \frac{4}{3!}t^3 + \frac{1}{5!}t^5$

3. $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{1}{s^2+2}\right)\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} - \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}$

$= \cos(\sqrt{2}t) - \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$.

4. $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\}$

$$\frac{s}{(s-2)(s-3)(s-6)} = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-6} = \frac{\frac{1}{2}}{s-2} - \frac{1}{s-3} + \frac{\frac{1}{2}}{s-6}$$

$s = A(s-3)(s-6) + B(s-2)(s-6) + C(s-2)(s-3).$

$s=2$ $\quad 2 = 4A \Rightarrow \frac{1}{2} = A$

$s=3$ $\quad 3 = -3B \Rightarrow -1 = B$

$s=6$ $\quad 6 = 12C \Rightarrow \frac{1}{2} = C$

So $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s-2} - \frac{1}{s-3} + \frac{\frac{1}{2}}{s-6}\right\}$

$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$

$= \frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}.$

5. $\mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s^2+4)}\right\}$

$$\frac{s}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4} = \frac{-\frac{1}{4}}{s+2} + \frac{\frac{1}{4}s + \frac{1}{2}}{s^2+4}$$

$s = A(s^2+4) + Bs(s+2) + C(s+2).$

$s=-2$ $\quad -2 = 8A \Rightarrow A = \frac{-1}{4}$

$s=0$ $\quad 0 = 4A + 2C = -1 + 2C \Rightarrow C = \frac{1}{2}$

$s=1$ $\quad 1 = 5A + 3B + 3C = 5\left(\frac{-1}{4}\right) + 3B + 3\left(\frac{1}{2}\right) \Rightarrow B = \frac{1}{4}$

$$\text{So } \mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s+2} + \frac{\frac{1}{4}s+\frac{1}{2}}{s^2+4}\right\}$$

$$= -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -\frac{1}{4}e^{-2t} + \frac{1}{4}\cos(2t) + \frac{1}{2}\mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{s^2+4}\right)\right\}$$

$$= -\frac{1}{4}e^{-2t} + \frac{1}{4}\cos(2t) + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -\frac{1}{4}e^{-2t} + \frac{1}{4}\cos(2t) + \frac{1}{4}\sin(2t).$$

6. $$\mathcal{L}^{-1}\left\{\frac{6s+3}{s^4+5s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{6s+3}{(s^2+1)(s^2+4)}\right\}$$

$$\frac{6s+3}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} = \frac{2s+1}{s^2+1} - \frac{2s+1}{s^2+4}$$

$$6s+3 = As(s^2+4) + B(s^2+4) + Cs(s^2+1) + D(s^2+1)$$

$s=0 \quad 3 = 4B + D$

$s=2i \quad 12i+3 = C(2i)(-3) + D(-3) = -6iC - 3D \implies C=-2$

$D=-1$

$B=1$

$s=i \quad 6i+3 = Ai(3) + 3B = 3iA + 3B \implies A = 2.$

$$\text{So } \mathcal{L}^{-1}\left\{\frac{6s+3}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1} - \frac{2s+1}{s^2+4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 2\cos(t) + \sin(t) - 2\cos(2t) - \mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{s^2+4}\right)\right\}$$

$$= 2\cos(t) + \sin(t) - 2\cos(t) - \frac{1}{2}\sin(2t).$$

Use the Laplace transform to solve the following IVPs.

7. $y'-y=2\cos(5t) \qquad y(0)=0$

$\mathcal{L}\{y'-y\}=\mathcal{L}\{2\cos(5t)\}$

$\mathcal{L}\{y'\}-\mathcal{L}\{y\}=2\mathcal{L}\{\cos(5t)\}$

$sY(s)-\cancel{y(0)}^{0}-Y(s)=\dfrac{2s}{s^2+25}$

$(s-1)Y(s)=\dfrac{2s}{s^2+25}$

$Y(s)=\dfrac{2s}{(s-1)(s^2+25)}$

$\dfrac{2s}{(s-1)(s^2+25)}=\dfrac{A}{s-1}+\dfrac{Bs+C}{s^2+25}=\dfrac{1}{13}\left[\dfrac{1}{s-1}-\dfrac{s}{s^2+25}+\dfrac{25}{s^2+25}\right]$

$2s=A(s^2+25)+Bs(s-1)+C(s-1)$

$s=1 \quad 2=26A \Rightarrow A=\dfrac{1}{13}$

$s=0 \quad 0=25A-C=\dfrac{25}{13}-C \qquad \Rightarrow C=\dfrac{25}{13}.$

$s=5i \quad 10i=B(5i)(5i-1)+C(5i-1)=-25B-5iB+5iC-C$

$\Rightarrow -25B-C=0$

$-25B-\dfrac{25}{13}=0 \quad \Rightarrow B=\dfrac{-1}{13}$

So $y(t)=\mathcal{L}^{-1}\{Y(s)\}$

$=\mathcal{L}^{-1}\left\{\dfrac{1}{13}\left[\dfrac{1}{s-1}-\dfrac{s}{s^2+25}+\dfrac{25}{s^2+25}\right]\right\}$

$=\dfrac{1}{13}\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}-\dfrac{1}{13}\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+25}\right\}+\dfrac{5}{13}\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+25}\right\}$

$=\dfrac{1}{13}e^t-\dfrac{1}{13}\cos(5t)+\dfrac{5}{13}\sin(5t)$

8. $y'' - 4y' = 6e^{3t} - 3e^{-t} \qquad y(0) = 1,\ y'(0) = -1$

$\mathcal{L}\{y'' - 4y'\} = \mathcal{L}\{6e^{3t} - 3e^{-t}\}$

$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} = 6\mathcal{L}\{e^{3t}\} - 3\mathcal{L}\{e^{-t}\}$

$$s^2Y(s) - s\overset{\to 1}{y(0)} - \overset{\to -1}{y'(0)} - 4(sY(s) - \overset{\to 1}{y(0)}) = \frac{6}{s-3} - \frac{3}{s+1}$$

$$(s^2 - 4s)Y(s) - s + 1 + 4 = \frac{6}{s-3} - \frac{3}{s+1}$$

$$s(s-4)Y(s) = \frac{6}{s-3} - \frac{3}{s+1} + s - 5$$

$$Y(s) = \frac{6}{s(s-3)(s-4)} - \frac{3}{s(s-4)(s+1)} + \frac{1}{s-4} - \frac{5}{s(s-4)}$$

$$= \frac{6(s+1) - 3(s-3) - 5(s-3)(s+1)}{s(s+1)(s-3)(s-4)} + \frac{1}{s-4}$$

$$= \frac{6s + 6 - 3s + 9 - 5(s^2 - 2s - 3)}{s(s+1)(s-3)(s-4)} + \frac{1}{s-4}$$

$$= \frac{-5s^2 + 13s + 30}{s(s-1)(s-3)(s-4)} + \frac{1}{s-4}$$

$$\frac{-5s^2 + 13s + 30}{s(s-1)(s-3)(s-4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3} + \frac{D}{s-4} = \frac{\frac{-5}{2}}{s} + \frac{\frac{16}{3}}{s-1} - \frac{\frac{4}{3}}{s-3} + \frac{\frac{1}{6}}{s-4}$$

$$-5s^2 + 13s + 30 = A(s-1)(s-3)(s-4) + Bs(s-3)(s-4) + Cs(s-1)(s-4) + Ds(s-1)(s-3)$$

$s = 0 \quad 30 = -12A \Rightarrow A = -\frac{5}{2}$

$s = 1 \quad 38 = 6B \Rightarrow B = \frac{16}{3}$

$s=3 \quad 24 = -18C \implies C = -\frac{4}{3}$

$s=4 \quad 2 = 12D \implies D = \frac{1}{6}$

So $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{ \frac{-\frac{5}{2}}{s} + \frac{\frac{16}{3}}{s-1} - \frac{\frac{4}{3}}{s-3} + \frac{\frac{1}{6}}{s-4} + \frac{1}{s-4} \right\}$$

$$= -\frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{16}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{7}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{5}{2} + \frac{16}{3}e^{t} + \frac{7}{6}e^{4t} - \frac{4}{3}e^{3t}$$