# 7.3 Operational Properties I

**Translation on s-axis**

Find $F(s)$ of the following

1. $\mathcal{L}\{te^{-6t}\} = \mathcal{L}\{t\}_{s \to s-(-6)}$

$$= \frac{1}{s^2}\Big|_{s \to s+6}$$

$$= \frac{1}{(s+6)^2}$$

2. $\mathcal{L}\{e^{2t}(t-1)^2\} = \mathcal{L}\{e^{2t}(t^2-2t+1)\}$

$$= \mathcal{L}\{t^2e^{2t}\} - 2\mathcal{L}\{te^{2t}\} + \mathcal{L}\{e^{2t}\}$$

$$= \mathcal{L}\{t^2\}_{s \to s-2} - 2\mathcal{L}\{t\}_{s \to s-2} + \mathcal{L}\{e^{2t}\}$$

$$= \left(\frac{2}{s^3}\right)\Big|_{s \to s-2} - 2\left(\frac{1}{s^2}\right)\Big|_{s \to s-2} + \frac{1}{s-2}$$

$$= \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}$$

3. $\mathcal{L}\{e^{-2t}\cos(4t)\} = \mathcal{L}\{\cos(4t)\}_{s \to s-(-2)}$

$$= \frac{s}{s^2+16}\Big|_{s \to s+2}$$

$$= \frac{s+2}{(s+2)^2+16}.$$

Find $f(t)$ of the following

4. $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}_{s \to s-1}$

$$= \mathcal{L}^{-1}\left\{\left(\frac{3!}{3!}\right)\left(\frac{1}{s^4}\right)\right\}_{s \to s-1}$$

$$= \frac{1}{6}t^3e^t.$$

5. $$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+5}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}_{s\to s+1}$$
$$=\mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{s^2+4}\right)\right\}_{s\to s+1}$$
$$=\frac{1}{2}e^{-t}\sin(2t).$$

6. $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+5}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}_{s\to s+2} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{s\to s+2}.$$
$$=e^{-2t}\cos(t) - 2e^{-2t}\sin(t).$$

Solve the following IVPs using the Laplace transform

7. $y' - y = 1 + te^t \qquad y(0) = 0$

$\mathcal{L}\{y' - y\} = \mathcal{L}\{1 + te^t\}$

$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\} + \mathcal{L}\{te^t\}$

$sY(s) - \overset{\nearrow 0}{y(0)} - Y(s) = \frac{1}{s} + \mathcal{L}\{t\}_{s \to s-1}$.

$(s-1)Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2}$

$Y(s) = \frac{1}{s(s-1)} + \frac{1}{(s-1)^3}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)} + \frac{1}{(s-1)^3}\right\}$.

$= \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}$

$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$

$1 = A(s-1) + Bs$

$s = 0 \quad 1 = -A \quad \Rightarrow A = -1$

$s = 1 \quad 1 = B$

$= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}_{s \to s-1}$

$= -1 + e^t + \mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{s^3}\right)\right\}_{s \to s-1}$

$= -1 + e^t + \frac{1}{2}t^2e^t$

8. $y'' - 6y' + 9y = t \qquad y(0) = 0,\ y'(0) = 1.$

$\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{t\}$

$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \dfrac{1}{s^2}$

$s^2 Y(s) - s\overset{\to 0}{y(0)} - \overset{\to 1}{y'(0)} - 6(sY(s) - \overset{\to 0}{y(0)}) + 9Y(s) = \dfrac{1}{s^2}$

$(s^2 - 6s + 9)Y(s) - 1 = \dfrac{1}{s^2}$

$(s-3)^2 Y(s) = \dfrac{1}{s^2} + 1.$

$Y(s) = \dfrac{1}{s^2(s-3)^2} + \dfrac{1}{(s-3)^2}.$

$\dfrac{1}{s^2(s-3)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-3} + \dfrac{D}{(s-3)^2} = \dfrac{\frac{2}{27}}{s} + \dfrac{\frac{1}{9}}{s^2} - \dfrac{\frac{2}{27}}{s-3} + \dfrac{\frac{1}{9}}{(s-3)^2}$

$1 = As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2$

$s=0 \qquad 1 = 9B \Rightarrow B = 1/9$

$s=3 \qquad 1 = 9D \Rightarrow D = 1/9$

$s=2 \qquad 1 = 2A + \frac{1}{9} - 4C + \frac{4}{9} \Rightarrow 2A - 4C = \frac{4}{9}$

$s=4 \qquad 1 = 4A + \frac{1}{9} + 16C + \frac{16}{9} \Rightarrow 4A + 16C = \frac{-8}{9}$

$4(2A - 4C = \frac{4}{9})$

$+ (4A + 16C = -8/9)$

$12A = \dfrac{8}{9} \Rightarrow A = \dfrac{2}{27}, \quad C = -\dfrac{2}{27}.$

So $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{ \frac{\frac{2}{27}}{s} + \frac{\frac{1}{9}}{s^2} - \frac{\frac{2}{27}}{s-3} + \frac{\frac{1}{9}}{(s-3)^2} + \frac{1}{(s-3)^2} \right\}$$

$$= \frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{10}{9}\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}_{s \to s-3}$$

$$= \frac{2}{27} + \frac{t}{9} - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}$$

$\boxed{\text{Translation on t-axis}}$

Find F(s) of the following.

9. $\mathcal{L}\{(t-1)\mathcal{U}(t-1)\}$

$\ulcorner$ $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$f(t) = t \Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$ $\lrcorner$

$$\mathcal{L}\{(t-1)\mathcal{U}(t-1)\} = \frac{e^{-s}}{s^2}$$

10. $\mathcal{L}\{e^{2-t}\mathcal{U}(t-2)\}$

$\ulcorner$ Alt Form: $\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$ $\lrcorner$

$$\mathcal{L}\{e^{2-t}\mathcal{U}(t-2)\} = e^{-2s}\mathcal{L}\{e^{2-(t+2)}\}$$

$$= e^{-2s}\mathcal{L}\{e^{-t}\}$$

$$= \frac{e^{-2s}}{s+1}.$$

11. $\mathcal{L}\{\sin(t)\mathcal{U}(t-\frac{\pi}{2})\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin(t+\frac{\pi}{2})\}$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos(t)\}$$

$$= \frac{e^{-\pi s}s}{s^2+1}.$$

12. $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 2 & 0 \le t < 3 \\ -2 & t \ge 3 \end{cases}$

$$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

$$= g(t)[\overbrace{\mathcal{U}(t-0)}^{=1} - \mathcal{U}(t-a)] + h(t)\mathcal{U}(t-a)$$

$$= g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

So $f(t) = 2 - 2\mathcal{U}(t-3) - 2\mathcal{U}(t-3) = 2 - 4\mathcal{U}(t-3)$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\{2 - 4\mathcal{U}(t-3)\}$

$$= 2\mathcal{L}\{1\} - 4\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$= \frac{2}{s} - 4\frac{e^{-3s}}{s}$$

13. $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 0 & 0 \le t < \frac{3\pi}{2} \\ \sin(t) & t \ge \frac{3\pi}{2} \end{cases}$

$f(t) = 0[\mathcal{U}(t-0) - \mathcal{U}(t-\frac{3\pi}{2})] + \sin(t)\mathcal{U}(t-\frac{3\pi}{2}) = \sin(t)\mathcal{U}(t-\frac{3\pi}{2})$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\mathcal{U}(t-\frac{3\pi}{2})\}$

$$= e^{-\frac{3\pi}{2}s}\mathcal{L}\{\sin(t+\frac{3\pi}{2})\}$$

$$= e^{-\frac{3\pi}{2}s}\mathcal{L}\{-\cos(t)\}$$

$$= -\frac{e^{-\frac{3\pi}{2}s}s}{s^2+1}$$

Find the inverse Laplace of the following.

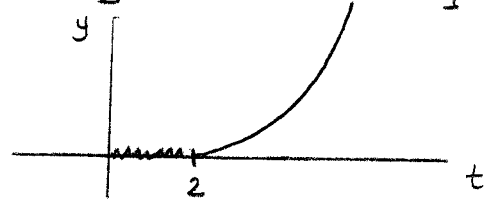
14. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$

$F(s) = \frac{1}{s^3}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = \frac{1}{2}t^2$

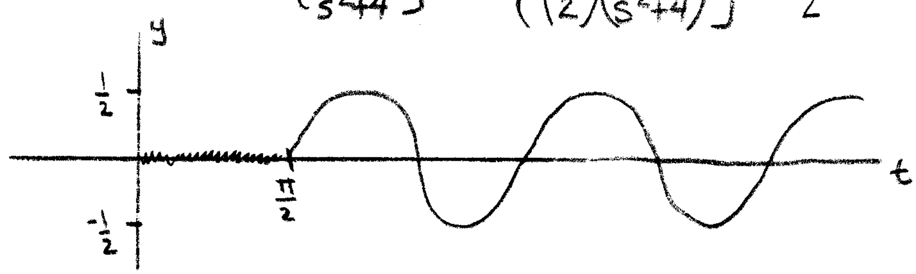
$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \frac{1}{2}(t-2)^2\,\mathcal{U}(t-2).$




15. $\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}s}{s^2+4}\right\} = \frac{1}{2}\sin\left(2\left(t-\frac{\pi}{2}\right)\right)\mathcal{U}\left(t-\frac{\pi}{2}\right) = -\frac{1}{2}\sin(t)\,\mathcal{U}\left(t-\frac{\pi}{2}\right)$

$F(s) = \frac{s}{s^2+4}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{s}{s^2+4}\right)\right\} = \frac{1}{2}\sin(2t)$



16. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\}$

$$F(s) = \frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}. = \frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s-1}$$

$$1 = As(s-1) + B(s-1) + Cs^2$$

$s=0 \quad 1 = -B \quad \Rightarrow B = -1.$

$s=1 \quad 1 = C$

$s=2 \quad 1 = 2A + B + 4C = 2A - 1 + 4 \quad \Rightarrow A = -1$

Then $f(t) = \mathcal{L}^{-1}\{F(s)\}$

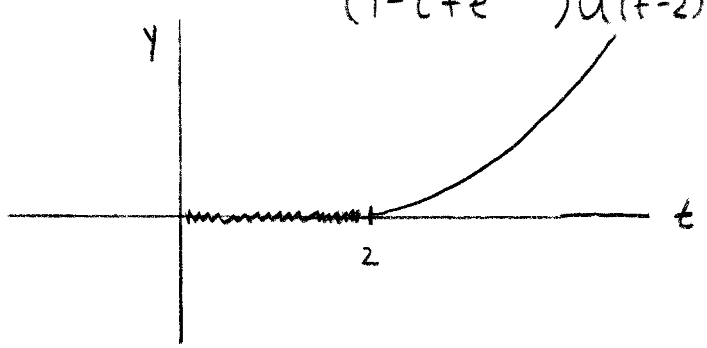
$$= \mathcal{L}^{-1}\left\{\frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s-1}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -1 - t + e^t$$

So $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\} = \left(-1 - (t-2) + e^{t-2}\right)\mathcal{U}(t-2)$

$$= (1 - t + e^{t-2})\,\mathcal{U}(t-2)$$



Solve the following IVPs using the Laplace transform.

17. $y' - y = f(t)$, $y(0) = 0$ where $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & t \ge 1. \end{cases}$

$$f(t) = 1 - \mathcal{U}(t-1) - \mathcal{U}(t-1) = 1 - 2\mathcal{U}(t-1)$$

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{1 - 2\mathcal{U}(t-1)\}$$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\} - 2\mathcal{L}\{\mathcal{U}(t-1)\}$$

$$sY(s) - \overset{\to 0}{y(0)} - Y(s) = \frac{1}{s} - \frac{2e^{-s}}{s}$$

$$(s-1)Y(s) = \frac{1}{s} - \frac{2e^{-s}}{s}$$

$$Y(s) = \frac{1}{s(s-1)} - \frac{2e^{-s}}{s(s-1)}$$

Note $F(s) = \dfrac{1}{s(s-1)} = \dfrac{A}{s} + \dfrac{B}{s-1} = \dfrac{-1}{s} + \dfrac{1}{s-1}$

$1 = A(s-1) + Bs$

$s = 0 \quad 1 = -A \quad \Rightarrow A = -1$

$s = 1 \quad B = 1$

Then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{-1}{s} + \dfrac{1}{s-1}\right\} = -1 + e^{t}$

So $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)} - \frac{2e^{-s}}{s(s-1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} - 2\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{s(s-1)}\right)\right\}$$

$$= -1 + e^{-t} - 2(-1 + e^{t-1})\mathcal{U}(t-1)$$

$$= \begin{cases} -1 + e^{-t} & 0 \le t < 1 \\ 1 + e^{-t} - 2e^{t-1} & t \ge 1. \end{cases}$$

18. $y'' + 4y = \sin(t)\mathcal{U}(t-2\pi) \qquad y(0)=1,\ y'(0)=0$

$\mathcal{L}\{y''+4y\} = \mathcal{L}\{\sin(t)\mathcal{U}(t-2\pi)\}$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = e^{-2\pi s}\mathcal{L}\{\sin(t-2\pi)\}$

$s^2Y(s) - s\overset{\to 1}{y(0)} - \overset{\to 0}{y'(0)} + 4Y(s) = e^{-2\pi s}\mathcal{L}\{\sin(t)\}$

$$(s^2+4)Y(s) - s = \frac{e^{-2\pi s}}{s^2+1}$$

$$Y(s) = \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$$

Note $F(s) = \dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}. = \dfrac{\frac{1}{3}}{s^2+1} - \dfrac{\frac{1}{3}}{s^2+4}$

$1 = As(s^2+4) + B(s^2+4) + Cs(s^2+1) + D(s^2+1)$

$s=i \quad 1 = 3iA + 3B \quad \Rightarrow A=0,\ B=\frac{1}{3}$

$s=2i \quad 1 = -3iC - 3D \quad \Rightarrow C=0,\ D=-\frac{1}{3}$

Then $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{s^2+4}\right)\right\}$$

$$= \frac{1}{3}\cos(t) - \frac{1}{6}\sin(2t)$$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= \left[\frac{1}{3}\cos(t-2\pi) - \frac{1}{6}\sin(2(t-2\pi))\right]\mathcal{U}(t-2\pi) + \cos(2t)$$

$$= \left[\frac{1}{3}\cos(t) - \frac{1}{6}\sin(2t)\right]\mathcal{U}(t-2\pi) + \cos(2t)$$