7-4 Operational Properties II.

Derivatives of a transform: Evaluate the following Laplace transforms

1. $\mathcal{L}\{te^{-10t}\} = -\frac{d}{ds}\mathcal{L}\{e^{-10t}\}$

$$= -\frac{d}{ds}\left[\frac{1}{s+10}\right]$$

$$= -\frac{\left[(s+10)(1') - (1)(s+10)'\right]}{(s+10)^2}$$

$$= \frac{1}{(s+10)^2}$$

Alternatively, $\mathcal{L}\{te^{-10t}\} = \mathcal{L}\{t\}_{s \to s-(-10)}$

$$= \left.\frac{1}{s^2}\right|_{s \to s+10}$$

$$= \frac{1}{(s+10)^2}$$

2. $\mathcal{L}\{t^2\cos(t)\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\cos(t)\}$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right].$$

$$= \frac{d}{ds}\left[\frac{-s^2+1}{(s^2+1)^2}\right]$$

$$= \frac{(s^2+1)^2(-2s) - (-s^2+1)(s^2+1)(2s)}{(s^2+1)^4}$$

$$= \frac{-2s(s^2+1) - 2s(-s^2+1)}{(s^2+1)^4}$$

$$= \frac{-2s}{(s^2+1)^4}$$

3 $\mathcal{L}\{te^{-3t}\cos(3t)\} = -\dfrac{d}{ds}\mathcal{L}\{e^{-3t}\cos(3t)\}$

$$= -\frac{d}{ds}\mathcal{L}\{\cos(3t)\}\Big|_{s\to s+3}$$

$$= -\frac{d}{ds}\left[\frac{s+3}{(s+3)^2+9}\right]$$

$$= -\frac{\left[((s+3)^2+9)(1) - (s+3)(2(s+3))\right]}{((s+3)^2+9)^2}$$

$$= -\frac{\left[-(s+3)^2+9\right]}{((s+3)^2+9)^2}$$

$$= \frac{(s+3)^2-9}{((s+3)^2+9)^2}$$

4. Solve the IVP $y'+y = t\sin(t)$, $y(0)$ using the Laplace transform

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{t\sin(t)\}$

$sY(s) - \overset{=0}{y(0)} + Y(s) = -\dfrac{d}{ds}\left[\dfrac{1}{s^2+1}\right] = \dfrac{2s}{(s^2+1)^2}$

$(s+1)Y(s) = \dfrac{2s}{(s^2+1)^2}$

$\Rightarrow Y(s) = \dfrac{2s}{(s+1)(s^2+1)^2} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1} + \dfrac{Ds+E}{(s^2+1)^2}$

$\Rightarrow 2s = A(s^2+1)^2 + Bs(s^2+1) + C(s+1)(s^2+1) + Ds(s+1) + E(s+1)$

$s=-1 \qquad -2 = 4A \Rightarrow A = -\frac{1}{2}$

$s=i \qquad 2i = Di(i+1) + E(i+1) = (-D+E) + i(D+E)$

$-D+E = 0 \qquad \Rightarrow E = 1$

$D+E = 2 \qquad D = 1.$

$s=0 \quad 0 = A + C + E$

$0 = \frac{-1}{2} + 1 + C \quad \Rightarrow C = -\frac{1}{2}$

$s=1 \quad 2 = 4A + 4B + 4C + 2D + 2E$

$2 = 4\left(\frac{-1}{2}\right) + 4B + 4\left(\frac{-1}{2}\right) + 2(1) + 2(1)$

$\Rightarrow B = \frac{1}{2}$

$$Y(s) = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}s - \frac{1}{2}}{s^2+1} + \frac{s+1}{(s^2+1)^2}$$

$$\text{So } y(t) = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{s}{(s^2+1)^2}\right)\right\} + \mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{(s^2+1)^2}\right)\right\}$$

$$= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) + \frac{1}{2}t\sin(t) + \frac{1}{2}\left(\sin(t) - t\cos(t)\right)$$

$$= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{t}{2}\sin(t) - \frac{t}{2}\cos(t)$$

Transforms of Integrals

Evaluate the following Laplace Transforms. Do not evaluate the integral before transforming.

5 $\mathcal{L}\{t^2 * te^t\} = \mathcal{L}\{t^2\}\mathcal{L}\{te^t\}$

$$= \left(\frac{2}{s^3}\right)\mathcal{L}\{t\}_{s \to s-1}$$

$$= \frac{2}{s^3}\left(\left.\frac{1}{s^2}\right|_{s \to s-1}\right)$$

$$= \frac{2}{s^3(s-1)^2}$$

Also could have been written as $\mathcal{L}\left\{\int_0^t \tau^2 (t-\tau)e^{t-\tau}\,d\tau\right\}$

6. $\mathcal{L}\{e^{2t} * \sin(t)\} = \mathcal{L}\{e^{2t}\}\mathcal{L}\{\sin(t)\}$

$$= \left(\frac{1}{s-2}\right)\left(\frac{1}{s^2+1}\right)$$

Also could have been written as $\mathcal{L}\{\int_0^t e^{2\tau}\sin(t-\tau)d\tau\}$

7. $\mathcal{L}\{\int_0^t \cos(\tau)d\tau\}$

$f(\tau) = \cos(\tau) \Rightarrow f(t) = \cos(t)$

$g(t-\tau) = 1 \Rightarrow g(t) = 1$

$$\mathcal{L}\{\int_0^t \cos(\tau)d\tau\} = \mathcal{L}\{\cos(t)\}\mathcal{L}\{1\}$$

$$= \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s}\right)$$

$$= \frac{1}{s^2+1}$$

8. $\mathcal{L}\{\int_0^t \tau\sin(\tau)d\tau\}$

$f(\tau) = \tau\sin(\tau) \Rightarrow f(t) = t\sin(t)$

$g(t-\tau) = 1 \Rightarrow g(t) = 1$

$$\mathcal{L}\{\int_0^t \tau\sin(\tau)d\tau\} = \mathcal{L}\{t\sin(t)\}\mathcal{L}\{1\}$$

$$= \left(\frac{2s}{(s^2+1)^2}\right)\left(\frac{1}{s}\right)$$

$$= \frac{2}{(s^2+1)^2}$$

9. $\mathcal{L}\{t\int_0^t \sin(\tau)d\tau\}$

$$f(\tau)=\sin(\tau) \Rightarrow f(t)=\sin(t)$$

$$g(t-\tau)=1 \Rightarrow g(t)=1.$$

$$\mathcal{L}\{t\int_0^t \sin(\tau)d\tau\} = -\frac{d}{ds}\mathcal{L}\{\int_0^t \sin(\tau)d\tau\}$$

$$= -\frac{d}{ds}\mathcal{L}\{\sin(t)\}\mathcal{L}\{1\}$$

$$= -\frac{d}{ds}\left[\left(\frac{1}{s^2+1}\right)\left(\frac{1}{s}\right)\right]$$

$$= -\frac{[(s^3+s)(0) - 1(3s^2+1)]}{(s^3+s)^2}$$

$$= \frac{3s^2+1}{(s^3+s)^2}$$

Evaluate the following inverse Laplace transforms.

10. $\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$

$$F(s)=\frac{1}{s} \qquad G(s)=\frac{1}{s-1}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= 1 * e^t$$

$$= e^t * 1$$

$$= \int_0^t e^\tau (1)d\tau$$

$$= e^\tau \Big|_0^t$$

$$= e^t - 1.$$

Note: Also could have beendone by partial fractions.

11. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$= t * e^t$

$= \int_0^t \tau e^{t-\tau} d\tau$

$= e^t \int_0^t \tau e^{-\tau} d\tau$

$u = \tau \qquad dv = e^{-\tau}$

$du = 1 \qquad v = -e^{-\tau}$

$= e^t\left[-\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau} d\tau\right]$

$= e^t\left[-te^{-t} - e^{-\tau}\Big|_0^t\right]$

$= e^t(-te^{-t} - e^{-t} + 1)$

$= -t - 1 + e^t$

Use the Laplace transform to solve the following integral equations.

12. $f(t) + \int_0^t (t-\tau) f(\tau) d\tau = t$

$\mathcal{L}\{f(t) + \int_0^t f(\tau)(t-\tau) d\tau\} = \mathcal{L}\{t\}$

$\mathcal{L}\{f(t)\} + \mathcal{L}\{\int_0^t f(t)(t-\tau) d\tau\} = \mathcal{L}\{t\}$

$F(s) + \mathcal{L}\{f(t)\}\mathcal{L}\{t\} = \frac{1}{s^2}$

$F(s) + \frac{F(s)}{s^2} = \frac{1}{s^2}$

$\frac{(s^2+1)F(s)}{s^2} = \frac{1}{s^2}$

$F(s) = \frac{1}{s^2+1}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$

13. $f(t) = te^t + \int_0^t \tau f(t-\tau)\,d\tau$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{te^t\} + \mathcal{L}\{\int_0^t f(\tau)(t-\tau)\,d\tau\}$ since $f*g = g*f$.

$F(s) = \mathcal{L}\{t\}_{s \to s-1} + \mathcal{L}\{f(t)\}\mathcal{L}\{t\}$

$$F(s) = \frac{1}{s^2}\Big|_{s-1} + \frac{F(s)}{s^2}$$

$$F(s) = \frac{1}{(s-1)^2} + \frac{F(s)}{s^2}$$

$$F(s) - \frac{F(s)}{s^2} = \frac{1}{(s-1)^2}$$

$$\frac{(s^2-1)F(s)}{s^2} = \frac{1}{(s-1)^2}$$

$$F(s) = \frac{s^2}{(s-1)^3(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+1}$$

$s^2 = A(s-1)^2(s+1) + B(s-1)(s+1) + C(s+1) + D(s-1)^3$

$s=1 \quad 1 = 2C \implies C = \frac{1}{2}$

$s=-1 \quad 1 = -8D \implies D = -\frac{1}{8}$

$s=0 \quad 0 = A - B + C - D = A - B + \frac{1}{2} + \frac{1}{8} \implies A - B = \frac{-5}{8}$

$s=2 \quad 4 = 3A + 3B + 3C + D = 3A + 3B + \frac{3}{2} - \frac{1}{8} \implies 3A + 3B = \frac{21}{8}$

$$A - B = \frac{-5}{8}$$
$$A + B = \frac{7}{8}$$

$$2A = \frac{2}{8} \implies A = \frac{1}{8},\ B = \frac{3}{4}$$

$$F(s) = \frac{\frac{1}{8}}{s-1} + \frac{\frac{3}{4}}{(s-1)^2} + \frac{\frac{1}{2}}{(s-1)^3} - \frac{\frac{1}{8}}{s+1}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{8}e^t + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}_{s \to s-1} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}_{s \to s-1} - \frac{1}{8}e^{-t}$$

$$= \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{2}\mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{s^3}\right)\right\}_{s \to s-1} - \frac{1}{8}e^{-t}$$

$$= \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t - \frac{1}{8}e^{-t}$$