7-5 Dirac Delta Function

1. $y'' + 16y = \delta(t-2\pi) \qquad y(0)=0,\ y'(0)=0$

$\mathcal{L}\{y'' + 16y\} = \mathcal{L}\{\delta(t-2\pi)\}$

$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = e^{-2\pi s}$

$(s^2+16)Y(s) = e^{-2\pi s}$

$Y(s) = e^{-2\pi s}\left(\frac{1}{s^2+16}\right)$

$F(s) = \frac{1}{s^2+16} \Rightarrow f(t) = \frac{1}{4}\sin(4t)$

$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+16}\right\} = \frac{1}{4}\sin(4(t-2\pi)\mathcal{U}(t-2\pi)$

$= \frac{1}{4}\sin(4t)\mathcal{U}(t-2\pi)$

2. $y'' + y = \delta(t-2\pi) + \delta(t-4\pi) \qquad y(0)=1,\ y'(0)=0$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2\pi)\} + \mathcal{L}\{\delta(t-4\pi)\}$

$s^2Y(s) - sy(0) - y'(0) + Y(s) = e^{-2\pi s} + e^{-4\pi s}$

$(s^2+1)Y(s) - s = e^{-2\pi s} + e^{-4\pi s}$

$(s^2+1)Y(s) = e^{-2\pi s} + e^{-4\pi s} + s$

$Y(s) = \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-4\pi s}}{s^2+1} + \frac{s}{s^2+1}$

$F(s) = \frac{1}{s^2+1} \Rightarrow f(t) = \sin(t)$.

$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-4\pi s}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$

$= \sin(t-2\pi)\mathcal{U}(t-2\pi) + \sin(t-4\pi)\mathcal{U}(t-4\pi) + \cos(t)$

$= \sin(t)\mathcal{U}(t-2\pi) + \sin(t)\mathcal{U}(t-4\pi) + \cos(t)$

3. $y'' - 7y' + 6y = e^t + \delta(t-2) + \delta(t-4)$. $\quad y(0) = y'(0) = 0$

$\mathcal{L}\{y'' - 7y' + 6y\} = \mathcal{L}\{e^t + \delta(t-2) + \delta(t-4)\}$

$s^2Y(s) - sy(0) - y'(0) - 7sY(s) + 7y(0) + 6Y(s) = \frac{1}{s-1} + e^{-2s} + e^{-4s}$

$(s^2 - 7s + 6)Y(s) = \frac{1}{s-1} + e^{-2s} + e^{-4s}$.

$Y(s) = \frac{1}{(s-1)^2(s-6)} + \frac{e^{-2s}}{(s-1)(s-6)} + \frac{e^{-4s}}{(s-1)(s-6)}$

$\frac{1}{(s-1)^2(s-6)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-6} = \frac{-1/5}{(s-1)^2} + \frac{1/5}{s-6}$

$1 = A(s-1)(s-6) + B(s-6) + C(s-1)$.

$s=1 \quad 1 = -5B \Rightarrow B = -1/5$.

$s=6 \quad 1 = 5C \Rightarrow C = 1/5$.

$s=0 \quad 1 = 6A - 6(-\frac{1}{5}) + \frac{1}{5}(-1) \Rightarrow A = 0$

$\frac{1}{(s-1)(s-6)} = \frac{A}{s-1} + \frac{B}{s-6} = \frac{-\frac{1}{5}}{s-1} + \frac{\frac{1}{5}}{s-6}$

$1 = A(s-6) + B(s-1)$

$s=1 \quad 1 = -5A \Rightarrow A = -\frac{1}{5}$

$s=6 \quad 1 = 5B \Rightarrow B = \frac{1}{5}$

$F(s) = \frac{-\frac{1}{5}}{(s-1)^2} + \frac{\frac{1}{5}}{s-6} \Rightarrow \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}}{s^2}\right\}_{s \to s-1} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} = -\frac{1}{5}e^t t + \frac{1}{5}e^{6t}$

$G(s) = \frac{-1/5}{s-1} + \frac{1/5}{s-6} \Rightarrow -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} = -\frac{1}{5}e^t + \frac{1}{5}e^{6t}$.

$y(t) = -\frac{1}{5}te^t + \frac{1}{5}e^{6t} - \frac{1}{5}e^{(t-2)}\mathcal{U}(t-2) + \frac{1}{5}e^{6(t-2)}\mathcal{U}(t-2) - \frac{1}{5}e^{(t-4)}\mathcal{U}(t-4)$

$+ \frac{1}{5}e^{6(t-4)}\mathcal{U}(t-4)$.