7.6 Systems of Linear DEs

1. $\frac{dx}{dt} = x - 2y$ $\quad \begin{cases} x(0) = -1 \\ y(0) = 2 \end{cases}$

$\frac{dy}{dt} = 5x - y$

$\begin{cases} x' - x + 2y = 0 \\ -5x + y' + y = 0 \end{cases}$

$\begin{cases} \mathcal{L}\{x' - x + 2y\} = 0 \\ \mathcal{L}\{-5x + y' + y\} = 0 \end{cases}$

$\begin{cases} sX(s) - x(0) - X(s) + 2Y(s) = 0 \\ -5X(s) + sY(s) - y(0) + Y(s) = 0 \end{cases}$

$\begin{cases} (s-1)X(s) + 2Y(s) + 1 = 0 \\ -5X(s) + (s+1)Y(s) - 2 = 0 \end{cases}$

$\begin{array}{l} 5 \\ (s-1) \end{array} \begin{cases} (s-1)X(s) + 2Y(s) = -1 \\ -5X(s) + (s+1)Y(s) = 2 \end{cases}$

$10Y(s) + (s^2 - 1)Y(s) = -5 + 2s - 2$

$(s^2 + 9)Y(s) = 2s - 7$

$Y(s) = \frac{2s}{s^2+9} - \frac{7}{s^2+9}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9} - \frac{7}{s^2+9}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - 7\mathcal{L}^{-1}\left\{\left(\frac{3}{3}\right)\left(\frac{1}{s^2+9}\right)\right\}$

$= 2\cos(3t) - \frac{7}{3}\sin(3t)$

$\begin{array}{l} (s+1) \\ -2 \end{array} \begin{cases} (s-1)X(s) + 2Y(s) = -1 \\ -5X(s) + (s+1)Y(s) = 2 \end{cases}$

$(s^2 - 1)X(s) + 10X(s) = -(s+1) - 4 = -s - 5$

$(s^2 + 9)X(s) = -s - 5$

$X(s) = \frac{-s-5}{(s^2+9)} = \frac{-s}{s^2+9} - \frac{5}{s^2+9}$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$
$$= -\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$
$$= -\cos(3t) - \frac{5}{3}\sin(3t).$$

$$x(t) = -\cos(3t) - \tfrac{5}{3}\sin(3t)$$
$$y(t) = 2\cos(3t) - \tfrac{7}{3}\sin(3t)$$

2. $\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = t^2$ $\quad\begin{cases} x(0)=8 \\ x'(0)=0 \end{cases}$ $\quad \begin{matrix} y(0)=0 \\ y'(0)=0. \end{matrix}$

$\frac{dx^2}{dt^2} - \frac{d^2y}{dt^2} = 4t$

$$\mathcal{L}\{x''+y''=t^2\}$$
$$\mathcal{L}\{x''-y''=4t\}$$

$$\begin{cases} s^2X(s) - sx(0) - x'(0) + s^2Y(s) - sy(0) - y'(0) = \frac{2}{s^3} \\ s^2X(s) - sx(0) - x'(0) - s^2Y(s) + sy(0) + y'(0) = \frac{4}{s^2} \end{cases}$$

$$\begin{cases} s^2X(s) + s^2Y(s) - 8s = \frac{2}{s^3} \\ s^2X(s) - s^2Y(s) - 8s = \frac{4}{s^2} \end{cases}$$

$$+.\quad\begin{cases} s^2X(s) + s^2Y(s) = \frac{2}{s^3} + 8s \\ s^2X(s) - s^2Y(s) = \frac{4}{s^2} + 8s. \end{cases}$$

$$2s^2X(s) = \frac{2}{s^3} + \frac{4}{s^2} + 16s$$

$$X(s) = \frac{1}{s^5} + \frac{2}{s^4} + \frac{8}{s}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$
$$= \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$
$$= \frac{1}{4!}t^4 + \frac{1}{3}t^3 + 8$$

$$(-)\begin{cases} s^2X(s) + s^2Y(s) = \frac{2}{s^3} + 8s \\ s^2X(s) - s^2Y(s) = \frac{4}{s^2} + 8s \end{cases}$$

$$2s^2Y(s) = \frac{2}{s^3} - \frac{4}{s^2}$$

$$Y(s) = \frac{1}{s^5} - \frac{2}{s^4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$
$$= \frac{1}{4!}t^4 - \frac{1}{3}t^3$$

$$x(t) = \frac{t^4}{4!} + \frac{t^3}{3} + 8$$

$$y(t) = \frac{t^4}{4!} - \frac{t^3}{3}$$

3. $$2\frac{dx}{dt}+\frac{dy}{dt}-2x=1 \qquad \begin{cases} x(0)=0 \\ y(0)=0 \end{cases}$$

$$\frac{dx}{dt}+\frac{dy}{dt}-3x-3y=2$$

$$\mathcal{L}\{2x'+y'-2x\}=\mathcal{L}\{1\}$$

$$\mathcal{L}\{x'+y'-3x-3y\}=\mathcal{L}\{2\}$$

$$\begin{cases} 2sX(s)-2\underset{0}{x(0)}+sY(s)-\underset{0}{y(0)}-2X(s)=\frac{1}{s} \\ sX(s)-s\underset{0}{x(0)}+sY(s)-\underset{0}{y(0)}-3X(s)-3Y(s)=\frac{2}{s} \end{cases}$$

$$\begin{array}{r l} (s-3) & \begin{cases} 2(s-1)X(s)+sY(s)=\frac{1}{s} \\ (s-3)X(s)+(s-3)Y(s)=\frac{2}{s} \end{cases} \\ -s & \end{array}$$

$$2(s-1)(s-3)X(s)-s(s-3)X(s)=\frac{s-3}{s}-2$$

$$(s-2)(s-3)X(s)=\frac{s-3}{s}-2$$

$$X(s)=\frac{s-3}{s(s-2)(s-3)}-\frac{2}{(s-2)(s-3)}\left(\frac{s}{s}\right)$$

$$=\frac{-s-3}{s(s-2)(s-3)}$$

$$=\frac{A}{s}+\frac{B}{s-2}+\frac{C}{s-3}$$

$$-s-3=A(s-2)(s-3)+Bs(s-3)+Cs(s-2)$$

$$s=0 \Rightarrow -3=6A \Rightarrow A=-\tfrac{1}{2}$$

$$s=2 \Rightarrow -5=-2B \Rightarrow B=\tfrac{5}{2}$$

$$s=3 \Rightarrow -6=3C \Rightarrow C=-2$$

$$=\frac{\frac{-1}{2}}{s}+\frac{\frac{5}{2}}{s-2}-\frac{2}{s-3}$$

Then $x(t) = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{5}{2}}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{-2}{s-3}\right\}$

$= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$= -\frac{1}{2} + \frac{5}{2}e^{2t} - 2e^{3t}$

$$\begin{array}{r} (s-3) \\ -2(s-1) \end{array}\left\{\begin{array}{l} 2(s-1)X(s) + sY(s) = \frac{1}{s} \\ (s-3)X(s) + (s-3)Y(s) = \frac{2}{s} \end{array}\right.$$

$s(s-3)Y(s) - 2(s-1)(s-3) = \frac{s-3}{s} - \frac{4(s-1)}{s} = -\frac{3s+1}{s}$

$-(s-2)(s-3)Y(s) = -\frac{3s+1}{s}$

$Y(s) = \frac{3s-1}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}$

$3s-1 = A(s-2)(s-3) + Bs(s-3) + Cs(s-2)$

$s=0 \Rightarrow -1 = 6A \Rightarrow A = -1/6$

$s=2 \Rightarrow 5 = -2B \Rightarrow B = -5/2$

$s=3 \Rightarrow 8 = 3C \Rightarrow C = 8/3$

$= \frac{-\frac{1}{6}}{s} - \frac{5/2}{s-2} + \frac{8/3}{s-3}$

Then $y(t) = -\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

$= -\frac{1}{6} - \frac{5}{2}e^{2t} + \frac{8}{3}e^{3t}$

Now $x(t) = -\frac{1}{2} + \frac{5}{2}e^{2t} - 2e^{3t}$

$y(t) = -\frac{1}{6} - \frac{5}{2}e^{2t} + \frac{8}{3}e^{3t}$

4. $\frac{dx}{dt} = y - \mathcal{U}(t - \frac{3\pi}{2})$ $\quad \begin{cases} x(0)=0 \\ y(0)=0 \end{cases}$

$\frac{dy}{dt} = -4x + 2\delta(t - \frac{3\pi}{2})$

$\mathcal{L}\{x' - y\} = -\mathcal{L}\{\mathcal{U}(t - \frac{3\pi}{2})\}$

$\mathcal{L}\{4x + y'\} = 2\mathcal{L}\{\delta(t - \frac{3\pi}{2})\}$

$$\begin{cases} sX(s) - \underset{0}{\underset{\parallel}{x(0)}} - Y(s) = -\dfrac{e^{-\frac{3\pi}{2}s}}{s} \\ 4X(s) + sY(s) - \underset{0}{\underset{\parallel}{y(0)}} = 2e^{-\frac{3\pi}{2}s} \end{cases}$$

$$\begin{array}{rl} s & \begin{cases} sX(s) - Y(s) = -\dfrac{e^{-\frac{3\pi}{2}s}}{s} \\ 4X(s) + sY(s) = 2e^{-\frac{3\pi}{2}s} \end{cases} \\ + & \\ \hline & (s^2+4)X(s) = -e^{-\frac{3\pi}{2}s} + 2e^{-\frac{3\pi}{2}s} = e^{-\frac{3\pi}{2}s} \end{array}$$

$\Longrightarrow X(s) = \dfrac{e^{-\frac{3\pi}{2}s}}{s^2+4}$

$F(s) = \dfrac{1}{s^2+4}$

$f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\left(\dfrac{2}{2}\right)\left(\dfrac{1}{s^2+4}\right)\right\} = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \dfrac{1}{2}\sin(2t)$

Then $x(t) = \dfrac{1}{2}\mathcal{L}^{-1}\left\{e^{-\frac{3\pi}{2}s}\left(\dfrac{2}{s^2+4}\right)\right\}$

$= \dfrac{1}{2}\sin(2(t - \frac{3\pi}{2}))\mathcal{U}(t - \frac{3\pi}{2})$

$= -\dfrac{1}{2}\sin(2t)\mathcal{U}(t - \frac{3\pi}{2})$

$$4\left\{\begin{array}{l} sX(s)-Y(s) = -\dfrac{e^{-\frac{3\pi}{2}s}}{s} \\ 4X(s)+sY(s) = 2e^{-\frac{3\pi}{2}s} \end{array}\right. \quad -s$$

$$-(s^2+4)Y(s) = -\frac{4e^{-3\pi/2 s}}{s} - 2se^{-3\pi/2 s}$$

$$= -\left(\frac{4+2s^2}{s}\right)e^{-\frac{3\pi}{2}s}$$

$$Y(s) = \left[\frac{4+2s^2}{s(s^2+4)}\right]e^{-\frac{3\pi}{2}s}$$

$$F(s) = \frac{4+2s^2}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{1}{s} + \frac{s}{s^2+4}$$

$$4+2s^2 = A(s^2+4)+Bs^2+Cs$$

$$s=0 \Rightarrow 4=4A \Rightarrow A=1$$

$$s=2i \Rightarrow 4+2(-4) = -4 = -4B+2iC \Rightarrow \begin{cases} B=1 \\ C=0 \end{cases}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = 1+\cos(2t)$$

$$\Rightarrow Y(s) = \left[\left(\frac{1}{s} + \frac{s}{s^2+4}\right)\right]e^{-\frac{3\pi}{2}s}$$

Then $y(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s} + \frac{s}{s^2+4}\right)e^{-\frac{3\pi}{2}s}\right\}$

$$= \left(1+\cos\left(2\left(t-\tfrac{3\pi}{2}\right)\right)\right)\mathcal{U}\left(t-\tfrac{3\pi}{2}\right)$$

$$= (1-\cos(2t))\,\mathcal{U}\left(t-\tfrac{3\pi}{2}\right)$$

So $\begin{cases} x(t) = -\frac{1}{2}\sin(2t)\,\mathcal{U}\left(t-\frac{3\pi}{2}\right) \\ y(t) = (1-\cos(2t))\,\mathcal{U}\left(t-\frac{3\pi}{2}\right) \end{cases}$