8.2 Homogeneous Linear Systems with Constant Coefficients

a. For each homogeneous linear system $\vec{x}' = A\vec{x}$, find the general solution.

b. Classify the critical point (0,0) by type and stability.

c. Draw the phase plane with at least 6 trajectories.

Real, distinct Eigenvalues

1. $\vec{x}' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$

a. Note: a solution to this system is of the form $\vec{x}(t) = \vec{K}_i e^{\lambda_i t}$ where λ_i is an eigenvalue of the given matrix (A) and $\vec{K}_i$ is its corresponding eigenvector.

So $0 = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix}$

$$= (-5-\lambda)(-2-\lambda) - 4$$
$$= 10 + 5\lambda + 2\lambda + \lambda^2 - 4$$
$$= \lambda^2 + 7\lambda + 6$$
$$= (\lambda+1)(\lambda+6)$$

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = -6$

For $\lambda_1 = -1$ we are looking for a vector $\vec{K}_1$ such that $\vec{K}_1$ satisfies $(A - \lambda_1 I)\vec{K}_1 = \vec{0}$

So $\left[\begin{array}{cc|c} -5-(-1) & 1 & 0 \\ 4 & -2-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -4 & 1 & 0 \\ 4 & -1 & 0 \end{array}\right]$

Note that the rows above are scalar multiples of each other. Recall that when the determinant of a matrix is 0, a row of that matrix can be

written as a linear combination of the other rows. This is the same as saying that the matrix is <u>not</u> invertible. If $\det(A-\lambda I) \neq 0$, then $\vec{K}_1 = (A-\lambda I)^{-1}\vec{0}$
$= \vec{0}$.

However, we are looking for nontrivial solutions that satisfy the homogeneous system.

Now we have $4k_1 - k_2 = 0$

$4k_1 = k_2$ 1 equation; 2 unknowns
Let k_1 be a free variable.

Let $k_1 = 1$. Then $\vec{K}_1 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 4k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

So $\vec{x}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$

For $\lambda_2 = -6$

$$\left[\begin{array}{cc|c} -5-(-6) & 1 & 0 \\ 4 & -2-(-6) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right].$$

$k_1 + k_2 = 0$

$-k_1 = k_2$ 1 equation, 2 unknowns
Let k_1 be FV with $k_1 = 1$

Then $\vec{K}_2 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

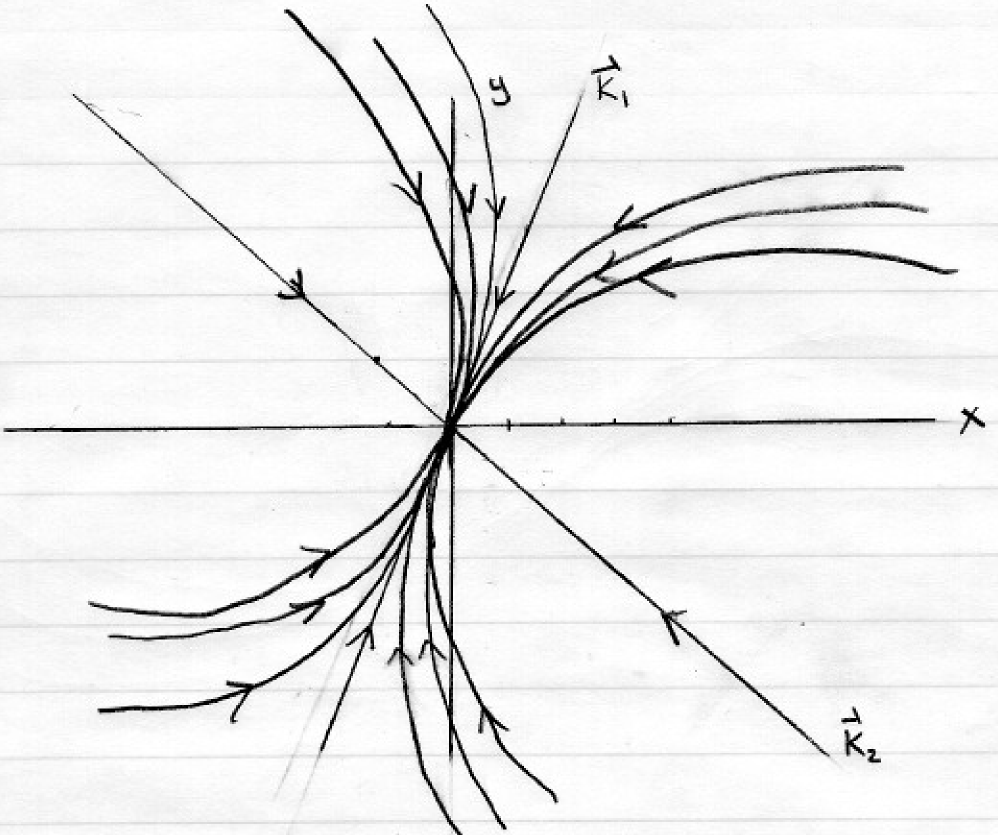
So $\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-6t}$

Then the general solution is $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$

$$= c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-6t}$$

b. $\left.\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -6 \end{array}\right\}$ Nodal sink, asymptotically stable.

c.



$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-6t}$$

$$\lim_{t \to \infty} \vec{x}(t) = \lim_{t \to \infty} e^{-t} \left(c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{(-6-(-1))t} \right)$$

$$\lim_{t \to \infty} e^{-t} = 0$$

$$\lim_{t \to \infty} \left(c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t} \right) = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$= \vec{0}$ along $\vec{K}_1$

For $\vec{x}' = A\vec{x}$, stability depends on the eigenvalues of A.

1. If any eigenvalue has a positive real part, then the critical point (zero/trivial solution) is <u>unstable</u>.
2. If all the eigenvalues with zero real parts are simple (multiplicity one) and all other eigenvalues have negative real parts, then the critical point is <u>stable</u> (on $[0,\infty)$ if the latter are also present).

3 If all eigenvalues have negative real parts, then the critical point is <u>asymptotically stable</u>.

2. $\vec{x}' = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \vec{x}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}$

$= (5-\lambda)(5-\lambda) - 9$

$= 25 - 5\lambda - 5\lambda + \lambda^2 - 9$

$= \lambda^2 - 10\lambda + 16$

$= (\lambda - 2)(\lambda - 8)$

$\Rightarrow \lambda_1 = 2,\ \lambda_2 = 8.$

For $\lambda_1 = 2$

$\left[\begin{array}{cc|c} 5-2 & 3 & 0 \\ 3 & 5-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right]$

$k_1 + k_2 = 0$

$-k_1 = k_2$ Let $k_1 = 1$ FV.

$\vec{K}_1 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

For $\lambda_2 = 8$

$\left[\begin{array}{cc|c} 5-8 & 3 & 0 \\ 3 & 5-8 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right]$

$-k_1 + k_2 = 0$

$k_1 = k_2$ Let $k_1 = 1$ FV

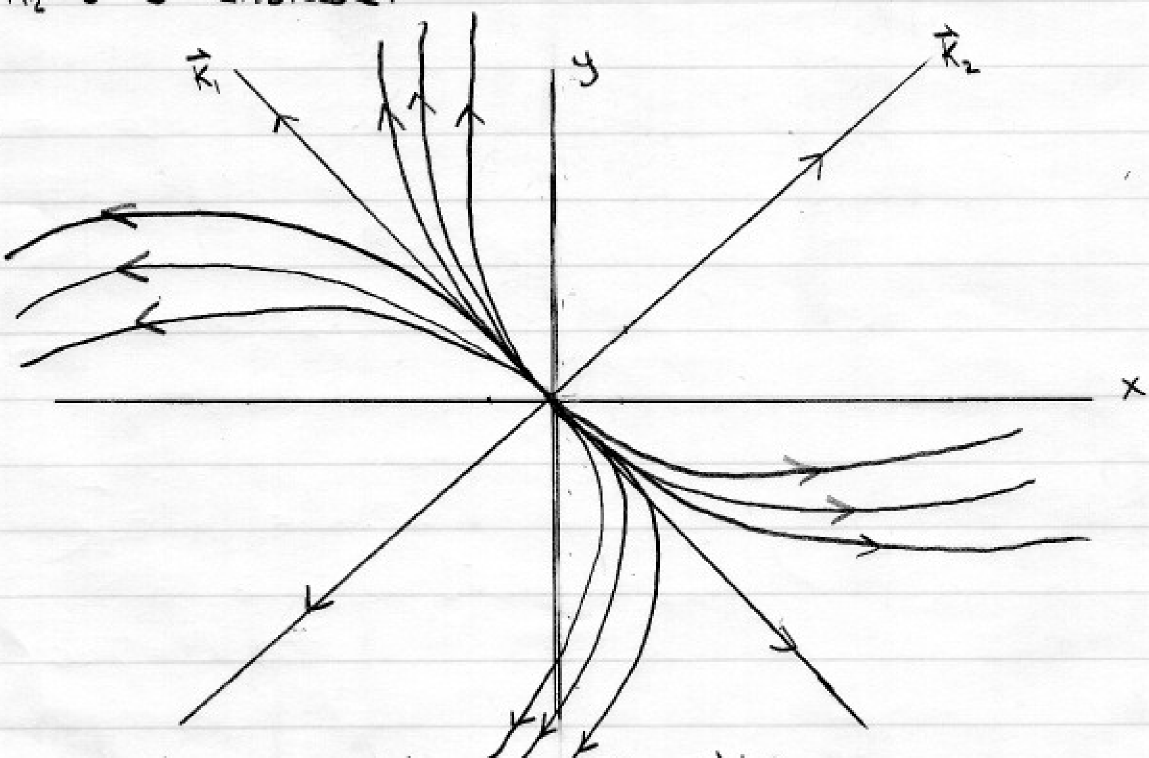
$\vec{K}_2 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}$

Then the general solution is $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}$

b. $\left.\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 8 \end{array}\right\}$ Nodal source unstable.

c.



Since we have a nodal source, $\lim_{t \to \infty} \vec{x}(t) = \infty$.

Or equivalently, $\lim_{t \to -\infty} \vec{x}(t) = \lim_{t \to -\infty} c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}$

$= \lim_{t \to -\infty} e^{2t} \left(c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{(8-2)t} \right)$

$$\lim_{t \to -\infty} e^{2t} = 0$$

$$\lim_{t \to -\infty} \left(c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} \right) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$= \vec{0}$ along $\vec{K}_1$

3. $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix}$

$$= (1-\lambda)(-2-\lambda) - 4$$
$$= -2 - \lambda + 2\lambda + \lambda^2 - 4$$
$$= \lambda^2 + \lambda - 6$$
$$= (\lambda+3)(\lambda-2).$$

$\Rightarrow \lambda_1 = -3, \ \lambda_2 = 2$

For $\lambda_1 = -3$

$$\left[\begin{array}{cc|c} 1-(-3) & 1 & 0 \\ 4 & -2-(-3) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right]$$

$4k_1 + k_2 = 0$

$-4k_1 = k_2$ Let $k_1 = 1$ FV

Then $\vec{K}_1 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -4k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

So $\vec{X}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$

For $\lambda_2=2$.

$$\left[\begin{array}{cc|c} 1-2 & 1 & 0 \\ 4 & -2-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$-k_1+k_2=0$$

$$k_1=k_2 \qquad \text{Let } k_1=1 \text{ FV}$$

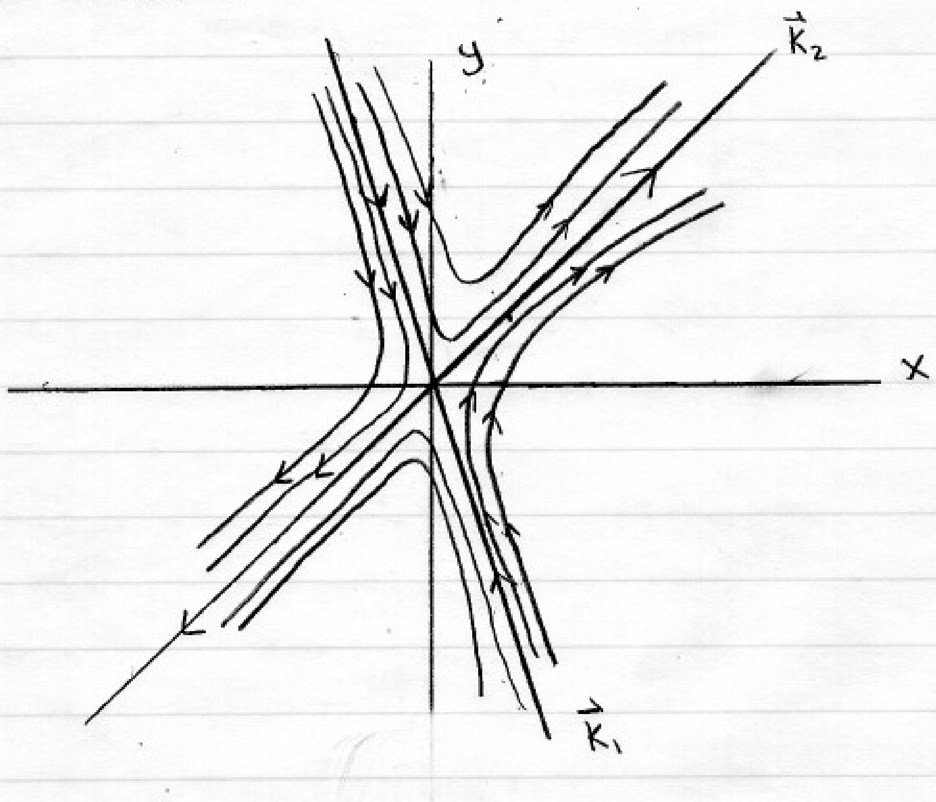
$$\vec{K}_2=\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}=\begin{bmatrix} k_1 \\ k_1 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $\vec{x}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$

Then the general solution is $\vec{x}(t)=c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-3t}+c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$

b. $\left.\begin{array}{l}\lambda_1=-3 \\ \lambda_2=2\end{array}\right\}$ Saddle point Unstable.

c.



Complex Eigenvalues

4. $\vec{x}\,' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{x}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix}.$

$$= (2-\lambda)(-2-\lambda) + 5$$
$$= -4 - 2\lambda + 2\lambda + \lambda^2 + 5$$
$$= \lambda^2 + 1$$

$\Rightarrow \lambda = \pm i$

Recall that if $\lambda_2 = \bar{\lambda}_1$, then $\vec{K}_2 = \overline{\vec{K}_1}$. Therefore we need only one eigenpair to get a general solution.

Let $\lambda = i$.

$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right].$ Note $5 = (2+i)(2-i)$

$k_1 - (2+i)k_2 = 0$

$k_1 = (2+i)k_2$ Let $k_2 = 1$. FV

$\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} (2+i)k_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

Then $\vec{z}(t) = \vec{K}e^{\lambda t}$

$$= \begin{bmatrix} 2+i \\ 1 \end{bmatrix} e^{it}$$

$$= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) (\cos(t) + i\sin(t)) \text{ by Euler's formula}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos(t) + i \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin(t) + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t)$$

Let $\vec{x}_1 = \text{Re}\,\vec{z}$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t)$$

$$= \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}$$

Let $\vec{x}_2 = \text{Im}\,\vec{z}$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t)$$

$$= \begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}.$$

Then the general solution is $\vec{x}(t) = c_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$

$^{\Gamma}$ we could have also chosen the other row $(2-i)k_1 - 5k_2 = 0$ to get $\vec{K} = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$.

Repeating the same process as before, this will give us a general solution of the form

$$\vec{x}(t) = c_1 \begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + c_2 \begin{bmatrix} 5\sin(t) \\ -\cos(t) + 2\sin(t) \end{bmatrix}.$$

$\perp$

b. $\lambda = 0 \pm i$ Center

Since the real part equals 0, the critical point is <u>stable</u>

c. Note that we have a pair of parametric equations

$$x(t) = c_1(2\cos(t) - \sin(t)) + c_2(2\sin(t) + \cos(t))$$

$$y(t) = c_1\cos(t) + c_2\sin(t).$$

Recall from Calc II such equations will give us ellipses center at the origin. The only thing left to do is find the direction of rotation.

<u>Determing the direction of rotation</u>

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

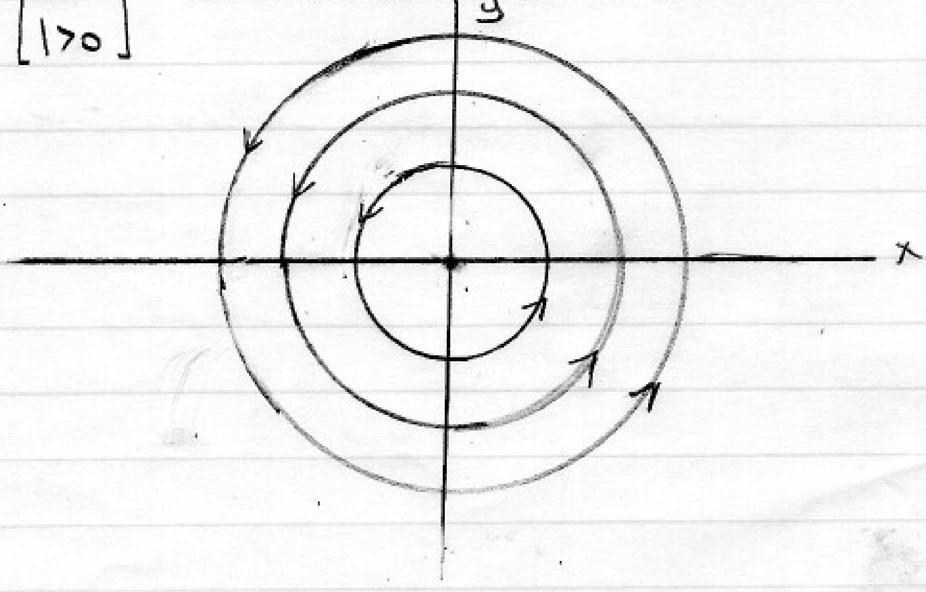
Then given $\vec{x}' = A\vec{x}$ pick the point $(1,0)$. To find the vector at this point, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

If $c > 0$ the vector points into the upper half-plane
$\Rightarrow$ rotation is counterclockwise.

If $c < 0$ the vector points into the lower half-plane
$\Rightarrow$ rotation is clockwise.

Now $\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 > 0 \end{bmatrix}$ counterclockwise



5. $\vec{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix}$

$= (-1-\lambda)^2 + 4$

$= 1 + \lambda + \lambda + \lambda^2 + 4$

$= \lambda^2 + 2\lambda + 5$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

Let $\lambda = -1 + 2i$

$\left[\begin{array}{cc|c} -1-(-1+2i) & -4 & 0 \\ 1 & -1-(-1+2i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right]$

$k_1 - 2ik_2 = 0$

$k_1 = 2ik_2$ Let $k_2 = 1$ FV

$\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2ik_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

Then $\vec{z} = \vec{K}e^{\lambda t}$

$= \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(-1+2i)t}$

$= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) e^{-t}(\cos(2t) + i\sin(2t))$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}\cos(2t) + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}\sin(2t) + i \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t}\sin(2t)$

Let $\vec{x}_1 = \operatorname{Re}\vec{z}$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t}\sin(2t)$$

$$= \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} e^{-t}$$

Let $\vec{x}_2 = \operatorname{Im}\vec{z}$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}\sin(2t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t}\cos(2t)$$

$$= \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix} e^{-t}$$

Then the general solution is $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$

$$= c_1 \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix} e^{-t}$$

b. $\lambda = -1 \pm 2i$ Spiral sink

Asymptotically stable.

c. Note that we have ~~the~~ pair of parametric equations

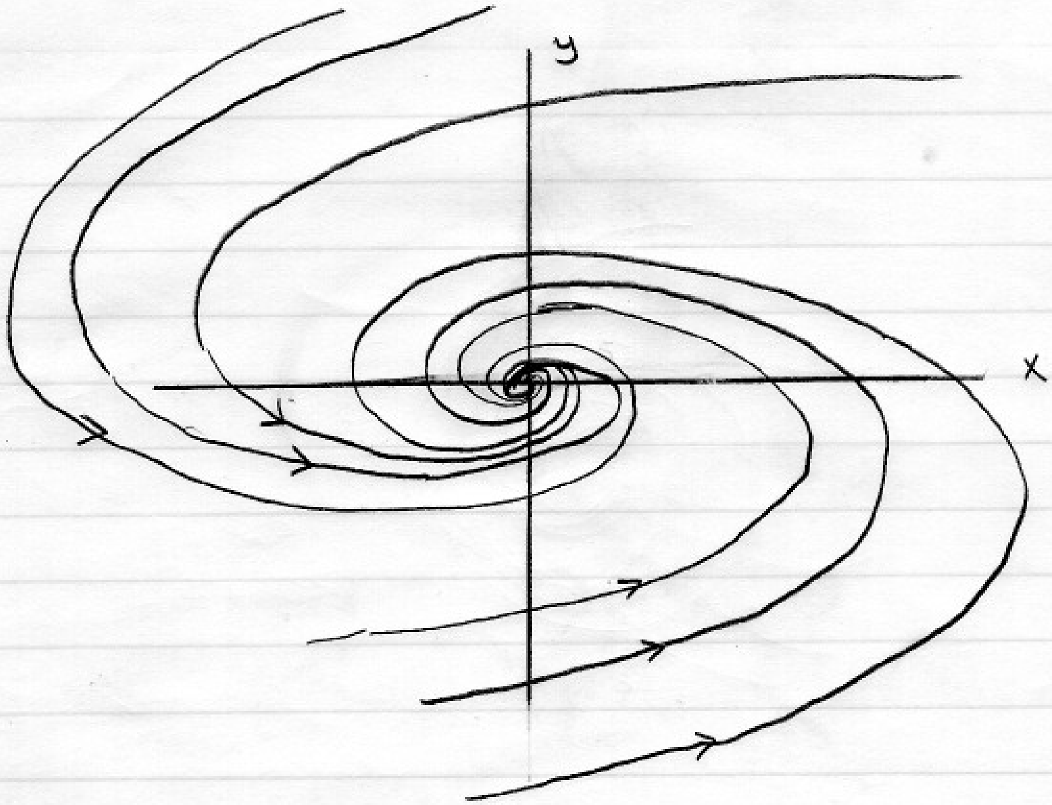
$x(t) = e^{-t}(-2c_1\sin(2t) + 2c_2\cos(2t))$

$y(t) = e^{-t}(c_1\cos(2t) + c_2\sin(2t))$

Here we have ellipses multiplied by an exponential gives a spiralling effect.

Then $\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 > 0 \end{bmatrix}$ counterclockwise.

c.



6. $\vec{x}' = \begin{bmatrix} -1 & 1 \\ -5 & 3 \end{bmatrix} \vec{x}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ -5 & 3-\lambda \end{vmatrix}$

$= (-1-\lambda)(3-\lambda) + 5$

$= -3 + \lambda - 3\lambda + \lambda^2 + 5$

$= \lambda^2 - 2\lambda + 2$

$\lambda = \dfrac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \dfrac{2 \pm 2i}{2} = 1 \pm i$

Let $\lambda = 1+i$

$\left[\begin{array}{cc|c} -1-(1+i) & 1 & 0 \\ -5 & 3-(1+i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -(2+i) & 1 & 0 \\ -5 & 2-i & 0 \end{array}\right]$ Note $5 = (2+i)(2-i)$

$-(2+i)k_1 + k_2 = 0$

$(2+i)k_1 = k_2$ Let $k_1 = 1$ FV

$$\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ (2+i)k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

Then $\vec{z} = \vec{K}e^{\lambda t}$

$$= \begin{bmatrix} 1 \\ 2+i \end{bmatrix} e^{(1+i)t}$$

$$= \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^t (\cos(t) + i\sin(t))$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t\cos(t) + i \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t\sin(t) + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t\sin(t)$$

Let $\vec{x}_1 = \operatorname{Re} \vec{z}$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t\sin(t)$$

$$= \begin{bmatrix} \cos(t) \\ 2\cos(t) - \sin(t) \end{bmatrix} e^t$$

Let $\vec{x}_2 = \operatorname{Im} \vec{z}$

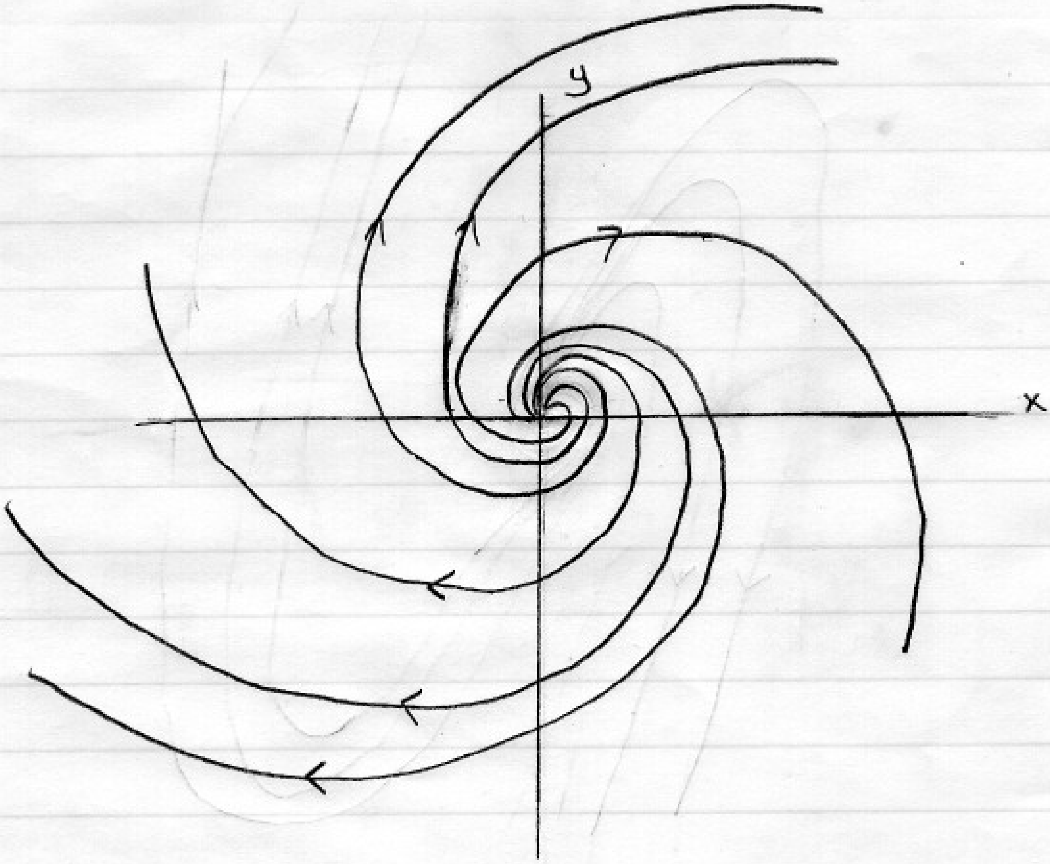
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t\sin(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t\cos(t)$$

$$= \begin{bmatrix} \sin(t) \\ 2\sin(t) + \cos(t) \end{bmatrix} e^t$$

b. $\lambda = 1 \pm i$ Spiral Source
Unstable.

c. $\begin{bmatrix} -1 & 1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 < 0 \end{bmatrix}$ clockwise.

c.



Repeated Eigenvalues

7. $\vec{x}' = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)^2$

$\Rightarrow \lambda_1 = 3 = \lambda_2$

For $\lambda = 3$

$\left[\begin{array}{cc|c} 3-3 & 0 & 0 \\ 0 & 3-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ Let $\left.\begin{array}{l} k_1 = s \\ k_2 = t \end{array}\right\}$ free variables

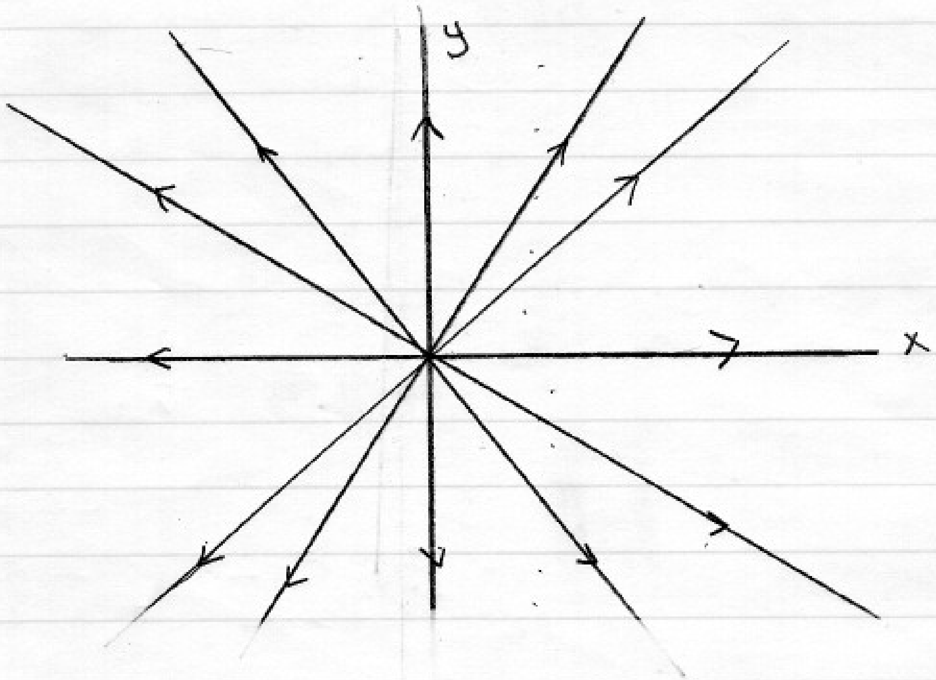
Then $\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{K}_1} + t \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{K}_2}$

So $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}$ $\quad$ $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$

and the general solution is $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$

b. $\lambda_1 = \lambda_2 = 3$ Proper (star) node (2 linearly independent eigenvectors)
Unstable.

c.



8. $\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix}$

$= (3-\lambda)(-1-\lambda) + 4$

$= -3 - 3\lambda + \lambda + \lambda^2 + 4$

$= \lambda^2 - 2\lambda + 1$

$= (\lambda - 1)^2$

$\Rightarrow \lambda_1 = 1 = \lambda_2$

For $\lambda = 1$ we have $(A - \lambda I)\vec{K} = \vec{0}$

$$\left[\begin{array}{cc|c} 3-1 & -4 & 0 \\ 1 & -1-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right].$$

$k_1 - 2k_2 = 0$

$k_1 = 2k_2$ Let $k_2 = 1$ FV

$$\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$

Only 1 linearly independent eigenvector. However, we need a 2nd linearly independent solution, so let

$$\vec{x}_2 = \vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t}$$

where $\vec{P}$ is the generalized eigenvector that satisfies $(A - \lambda I)\vec{P} = \vec{K}$.

So $\left[\begin{array}{cc|c} 3-1 & -4 & 2 \\ 1 & -1-1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right]$

$p_1 - 2p_2 = 1$

$p_1 = 1 + 2p_2$

$$P = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 + 2p_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Note: the second term is just a scalar multiple of $\vec{K}$, so when we write out the general solution,

$$\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) e^t,$$

part of our second solution will be aborbed into the first. Then $\vec{x}(t) = c_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$

Let $p_2 = 0$, then $\vec{P} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{x}_2 = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$

So the general solution is $\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$

b. $\lambda_1 = \lambda_2 = 1$ Improper (degenerate) node (only 1 linearly independent eigenvector).
Unstable.

c. Our general solution is of the form

$$\vec{x}(t) = c_1 \vec{K} e^t + c_2 \vec{K} t e^t + c_2 \vec{P} e^{\lambda t}$$
$$= te^t \left(\frac{c_1 \vec{K}}{t} + c_2 \vec{K} + \frac{c_2 \vec{P}}{t} \right)$$

Then $\lim_{t \to \infty} \vec{x}(t) = \lim_{t \to \infty} te^t \left(\frac{c_1 \vec{K}}{t} + c_2 \vec{K} + \frac{c_2 \vec{P}}{t} \right)$

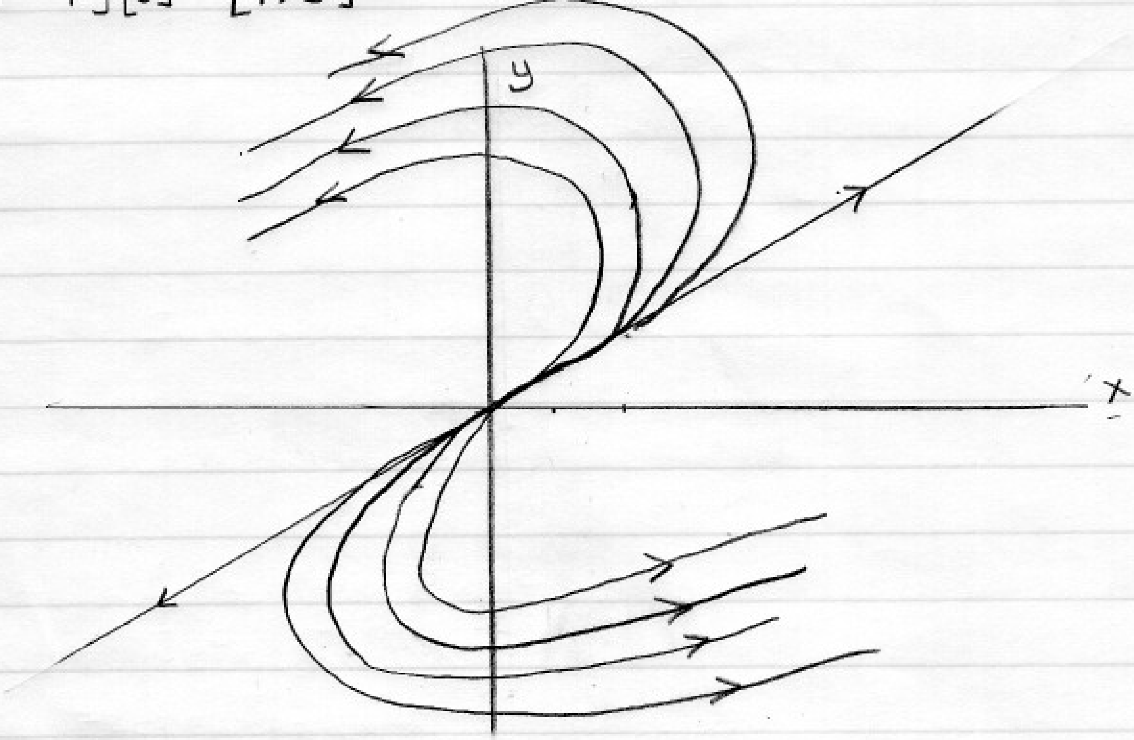
$$\lim_{t \to \infty} te^t = \infty$$
$$\lim_{t \to \infty} \left(\frac{c_1 \vec{K}}{t} + c_2 \vec{K} + \frac{c_2 \vec{P}}{t} \right) = c_2 \vec{K}$$

$= \infty$

So the trajectories move away from the origin along $\vec{K}$ in opposite directions.

To determine direction, we use the same trick as in the complex eigenvalues case.

$$\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1>0 \end{bmatrix}$$ counterclockwise.



9. $\vec{x}' = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}\vec{x}$

a. $0 = \det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix}$

$= (-6-\lambda)(4-\lambda) + 25$

$= -24 + 6\lambda - 4\lambda + \lambda^2 + 25$

$= \lambda^2 + 2\lambda + 1$

$= (\lambda + 1)^2$

$\Rightarrow \lambda_1 = \lambda_2 = -1$

Next we need to find a vector $\vec{K}$ such that $(A-\lambda I)\vec{K} = \vec{0}$.

$$\left[\begin{array}{cc|c} -6-(-1) & 5 & 0 \\ -5 & 4-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -5 & 5 & 0 \\ -5 & 5 & 0 \end{array}\right]$$

$-k_1 + k_2 = 0$

$k_1 = k_2$ Let $k_1 = 1$ FV.

Then $\vec{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

Now we need to find a vector $\vec{P}$ such that $(A-\lambda I)\vec{P} = \vec{K}$,

$$\left[\begin{array}{cc|c} -6-(-1) & 5 & 1 \\ -5 & 4-(-1) & 1 \end{array}\right] = \left[\begin{array}{cc|c} -5 & 5 & 1 \\ -5 & 5 & 1 \end{array}\right]$$

$-5p_1 + 5p_2 = 1$

$p_2 = \frac{1}{5} + p_1$

Then $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ \frac{1}{5} + p_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} + p_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $p_1 = 0 \Rightarrow \vec{P} = \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix}$

So $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} e^{-t}$

and the general solution is $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} \right) e^{-t}$

b. $\lambda_1 = \lambda_2 = -1$ Improper (degenerate) node (only 1 linearly independent eigenvector)
Asymptotically stable.

c. $\vec{x}(t) = c_1\vec{K}e^{-t} + c_2(\vec{K}te^{-t} + \vec{P}e^{-t})$

$$= te^{-t}\left(\frac{c_1\vec{K}}{t} + c_2\vec{K} + \frac{c_2\vec{P}}{t}\right)$$

Then $\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty} te^{-t}\left(\frac{c_1\vec{K}}{t} + c_2\vec{K} + \frac{c_2\vec{P}}{t}\right)$

$$\lim_{t\to\infty} te^{-t} = 0$$

$$\lim_{t\to\infty}\left(\frac{c_1\vec{K}}{t} + c_2\vec{K} + \frac{c_2\vec{P}}{t}\right) = c_2\vec{K}$$

$= \vec{0}$

Now the trajectories of $\vec{x}(t)$ approach $\vec{0}$ along $\vec{K}$ from opposite directions

To determine direction, we take $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -5<0 \end{bmatrix}$ clockwise