8.3 Nonhomogeneous Linear Systems

In the previous section, we solved homogeneous linear systems of the form

$$\vec{x}' = A\vec{x} \qquad (1)$$

Now we will look at nonhomegeneous linear systems of the form

$$\vec{x}' = A\vec{x} + \vec{F}(t) \qquad (2)$$

where $\vec{F}(t)$ is called a "vector-valued" function.

In this section, we will see how the nonhomogeneous techniques we learned in Chapter 4 work when there is more than one equation.

Part 1: Method of Undetermined Coefficients

1. $$\vec{x}' = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{t}$$

First we need to solve the homogeneous system.
From problem 2 in 8.2, we have

$$\vec{x}_h = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{8t}$$

Note that $\vec{F}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{t}$

Since $\vec{x}_p$ is the solution that satisfies the nonhomogeneous system, we expect $\vec{x}_p$ will have a similar form as $\vec{F}(t)$. So let's suppose $\vec{x}_p = \begin{bmatrix} a \\ b \end{bmatrix} e^t$

where a, b are constants.

Next we'll plug $\vec{x}_p$ into the nonhomogeneous system to find a, b such that $\vec{x}_p$ satisfies (2).

LHS: $\vec{x}_p\,' = \begin{bmatrix} a \\ b \end{bmatrix} e^t$

RHS: $$A\vec{x}_p + \vec{F}(t) = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\left(\begin{bmatrix} a \\ b \end{bmatrix} e^t\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$
$$= \begin{bmatrix} 5a+3b \\ 3a+5b \end{bmatrix} e^t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$
$$= \begin{bmatrix} 5a+3b+1 \\ 3a+5b-1 \end{bmatrix} e^t$$

Now we have $\begin{bmatrix} a \\ b \end{bmatrix} e^t = \begin{bmatrix} 5a+3b+1 \\ 3a+5b-1 \end{bmatrix} e^t$

Rewritten, we have $\begin{bmatrix} -4a-3b \\ -3a-4b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow$ 2 equations, 2 unknowns

Then
$$\begin{aligned} &3(4a+3b=-1) \\ &\underline{-4(3a+4b=1)} \\ &\quad -7b=-7 \quad \Rightarrow b=1,\ a=-1 \end{aligned}$$

So $\vec{x}_p = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t$.

and
$$\vec{x}(t) = \vec{x}_h + \vec{x}_p = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t$$

Now let's change the problem slightly...

2. $\vec{x}' = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\vec{x} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{2t}$

As before, $\vec{x}_h = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}$

Note that $\vec{F}(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{2t}$

Notice that $\vec{F}(t)$ has the same exponential as one of our components to $\vec{x}_h$. Therefore we can say there is repeated eigenvalue of 2. From section 8.2 on repeated eigenvalues, we can assume

$$\vec{x}_p = \begin{bmatrix} a \\ b \end{bmatrix} te^{2t} + \begin{bmatrix} c \\ d \end{bmatrix} e^{2t}$$

Plugging $\vec{x}_p$ into the equation, we have

LHS: $\vec{x}_p' = \begin{bmatrix} a \\ b \end{bmatrix} e^{2t} + \begin{bmatrix} 2a \\ 2b \end{bmatrix} te^{2t} + \begin{bmatrix} 2c \\ 2d \end{bmatrix} e^{2t}$

$$= \begin{bmatrix} 2a \\ 2b \end{bmatrix} te^{2t} + \begin{bmatrix} a+2c \\ b+2d \end{bmatrix} e^{2t}$$

RHS: $A\vec{x}_p + \vec{F}(t) = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\left(\begin{bmatrix} a \\ b \end{bmatrix} te^{2t} + \begin{bmatrix} c \\ d \end{bmatrix} e^{2t}\right) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{2t}$

$$= \begin{bmatrix} 5a+3b \\ 3a+5b \end{bmatrix} te^{2t} + \begin{bmatrix} 5c+3d+2 \\ 3c+5d+3 \end{bmatrix} e^{2t}$$

Now $\begin{bmatrix} 2a \\ 2b \end{bmatrix} te^{2t} + \begin{bmatrix} a+2c \\ b+2d \end{bmatrix} e^{2t} = \begin{bmatrix} 5a+3b \\ 3a+5b \end{bmatrix} te^{2t} + \begin{bmatrix} 5c+3d+2 \\ 3c+5d+3 \end{bmatrix} e^{2t}$

Matching terms on the left and right, we get the following equations

$$2a = 5a + 3b$$
$$2b = 3a + 5b$$
$$a + 2c = 5c + 3d + 2$$
$$b + 2d = 3c + 5d + 3$$

or in system form

$$\left[\begin{array}{cccc|c} 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 1 & 0 & -3 & -3 & 2 \\ 0 & 1 & -3 & -3 & 3 \end{array}\right].$$

Now after row reducing we have

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 1 & -5/6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} a = -\frac{1}{2} \\ b = \frac{1}{2} \\ c + d = -\frac{5}{6} \end{array}$$

Let $d = k$ be our free variable

Then $\vec{x}_p = \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} -\frac{5}{6} - k \\ k \end{bmatrix} e^{2t}$

$$= \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} - \frac{5}{6}\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} \underbrace{- k\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}}_{\text{absorbed into } \vec{x}_h}$$

$$= \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} - \frac{5}{6}\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

So $\vec{x}(t) = \vec{x}_h + \vec{x}_p$

$$= c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} - \frac{5}{6}\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

Part 2: Variation of Parameters

3. $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{2t}$

From problem 3 in 8.2, $\vec{x}_h = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

$$= \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \Phi(t)\vec{c}$$

where $\Phi(t)$ is called a <u>fundamental matrix</u>.

Recall $\vec{x}_p = \Phi(t)\int \Phi^{-1}(t)\vec{F}(t)\,dt$

If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now $\det \Phi(t) = \begin{vmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{vmatrix} = e^{-t} + 4e^{-t} = 5e^{-t}$

So $\Phi^{-1}(t) = \frac{e^{t}}{5} \begin{bmatrix} e^{2t} & -e^{2t} \\ 4e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{bmatrix}$

Then $\vec{x}_p = \Phi(t)\int \Phi^{-1}(t)\vec{F}(t)\,dt$

$$= \frac{1}{5}\Phi(t)\int \begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{2t}\,dt$$

$$= \frac{1}{5}\Phi(t)\int \begin{bmatrix} e^{5t} + 5e^{5t} \\ 4 - 5 \end{bmatrix} dt$$

$$= \frac{1}{5}\Phi(t)\int\begin{bmatrix}6e^{5t}\\-1\end{bmatrix}dt$$

$$= \frac{1}{5}\begin{bmatrix}e^{-3t} & e^{2t}\\-4e^{-3t} & e^{2t}\end{bmatrix}\begin{bmatrix}\frac{6}{5}e^{5t}\\-t\end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix}6e^{2t}-5te^{2t}\\-24e^{2t}-5te^{2t}\end{bmatrix}$$

$$= \frac{6}{25}\begin{bmatrix}1\\-4\end{bmatrix}e^{2t} - \frac{1}{5}\begin{bmatrix}1\\1\end{bmatrix}te^{2t}$$

Now $\vec{x}(t) = \vec{x}_h + \vec{x}_p$

$$= \Phi(t)\vec{c} + \Phi(t)\int\Phi^{-1}(t)\vec{F}(t)\,dt$$

$$= c_1\begin{bmatrix}1\\-4\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t} - \frac{1}{5}\begin{bmatrix}1\\1\end{bmatrix}te^{2t} + \frac{6}{25}\begin{bmatrix}1\\-4\end{bmatrix}e^{2t}$$

4. $\vec{x}' = \begin{bmatrix}2 & -5\\1 & -2\end{bmatrix}\vec{x} + \begin{bmatrix}\sec(t)\\0\end{bmatrix}$

From problem 4 in 8.2, $\vec{x}_h = c_1\begin{bmatrix}5\cos(t)\\2\cos(t)+\sin(t)\end{bmatrix} + c_2\begin{bmatrix}5\sin(t)\\-\cos(t)+2\sin(t)\end{bmatrix}$

$$= \begin{bmatrix}5\cos(t) & 5\sin(t)\\2\cos(t)+\sin(t) & -\cos(t)+2\sin(t)\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$= \Phi(t)\vec{c}$$

$$\det\Phi(t) = \begin{vmatrix}5\cos(t) & 5\sin(t)\\2\cos(t)+\sin(t) & -\cos(t)+2\sin(t)\end{vmatrix}$$

$$= -5\cos^2(t) + 10\cos(t)\sin(t) - 10\cos(t)\sin(t) - 5\sin^2(t)$$

$$= -5$$

$$\Phi^{-1}(t) = \frac{-1}{5}\begin{bmatrix} -\cos(t)+2\sin(t) & -5\sin(t) \\ -2\cos(t)-\sin(t) & 5\cos(t) \end{bmatrix}$$

Now $\vec{x}_p = \Phi(t)\int \Phi^{-1}(t)\vec{F}(t)\,dt$

$$= \frac{-1}{5}\Phi(t)\int \begin{bmatrix} -\cos(t)+2\sin(t) & -5\sin(t) \\ -2\cos(t)-\sin(t) & 5\cos(t) \end{bmatrix}\begin{bmatrix} \sec(t) \\ 0 \end{bmatrix} dt$$

$$= \frac{-1}{5}\Phi(t)\int \begin{bmatrix} -1+2\tan(t) \\ -2-\tan(t) \end{bmatrix} dt$$

$$= \frac{-1}{5}\begin{bmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t)+\sin(t) & -\cos(t)+2\sin(t) \end{bmatrix}\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\ln|\cos(t)\right)$$

$$= \frac{-1}{5}\begin{bmatrix} -5\cos(t)-10\sin(t) \\ -2\cos(t)-\sin(t)+2\cos(t)-4\sin(t) \end{bmatrix} t$$

$$\frac{-1}{5}\begin{bmatrix} -10\cos(t)+5\sin(t) \\ -4\cos(t)-2\sin(t)-\cos(t)+2\sin(t) \end{bmatrix}\ln|\cos(t)|$$

$$= \frac{-1}{5}\begin{bmatrix} -5\cos(t)-10\sin(t) \\ -5\sin(t) \end{bmatrix} t - \frac{1}{5}\begin{bmatrix} -10\cos(t)+5\sin(t) \\ -5\cos(t) \end{bmatrix}\ln|\cos(t)|$$

$$= \begin{bmatrix} \cos(t)+2\sin(t) \\ \sin(t) \end{bmatrix} t + \begin{bmatrix} 2\cos(t)-\sin(t) \\ \cos(t) \end{bmatrix}\ln|\cos(t)|.$$

So $\vec{x}(t) = \vec{x}_h + \vec{x}_p$

$$= c_1\begin{bmatrix} 5\cos(t) \\ 2\cos(t)+\sin(t) \end{bmatrix} + c_2\begin{bmatrix} 5\sin(t) \\ -\cos(t)+2\sin(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \cos(t)+2\sin(t) \\ \sin(t) \end{bmatrix} t + \begin{bmatrix} 2\cos(t)-\sin(t) \\ \cos(t) \end{bmatrix}\ln|\cos(t)|$$