3-1 § 1-8, 11, 12, 17, 19, 21, 23.

1. $y''+2y'-3y=0$
$r^2+2r-3=0$
$(r+3)(r-1)=0$
$r_1=-3,\ r_2=1$
$y=c_1e^{-3t}+c_2e^{t}$

2. $y''+3y'+2y=0$
$r^2+3r+2=0$
$(r+2)(r+1)=0$
$r_1=-2,\ r_2=-1$
$y=c_1e^{-2t}+c_2e^{-t}$

3. $6y''-y'-y=0$
$6r^2-r-1=0$
$r=\frac{-(-1)\pm\sqrt{1-4(6)(-1)}}{12}=\frac{1\pm\sqrt{25}}{12}$
$y=c_1e^{\frac{1}{2}t}+c_2e^{-\frac{1}{3}t}$

4. $2y''-3y'+y=0$
$2r^2-3r+1=0$
$r=\frac{3\pm\sqrt{9-4(2)(1)}}{4}$
$r_1=1,\ r_2=\frac{1}{2}$
$y=c_1e^{t}+c_2e^{\frac{1}{2}t}$

5. $y''+5y'=0$
$r^2+5r=0$
$r(r+5)=0$
$r_1=-5\ \ r_2=0$
$y=c_1e^{-5t}+c_2$

6. $4y''-9y=0$
$4r^2-9=0$
$(2r+3)(2r-3)=0$
$r_1=-\frac{3}{2}\quad r_2=\frac{3}{2}$
$y=c_1e^{-3/2t}+c_2e^{3/2t}$

7. $y''-9y'+9y=0$
$r^2-9r+9=0$
$r=\frac{9\pm\sqrt{81-4(9)(1)}}{2(1)}$
$r_1=\frac{9+3\sqrt{5}}{2}\quad r_2=\frac{9-3\sqrt{5}}{2}$
$y=c_1e^{\left(\frac{9+3\sqrt{5}}{2}\right)t}+c_2e^{\left(\frac{9-3\sqrt{5}}{2}\right)t}$

8. $y''-2y'-2y=0$
$r^2-2r-2=0$
$r=\frac{2\pm\sqrt{4-4(1)(-2)}}{2}=\frac{2\pm\sqrt{12}}{2}$
$r_1=1+\sqrt{3}\quad r_2=1-\sqrt{3}$
$y=c_1e^{(1+\sqrt{3})t}+c_2e^{(1-\sqrt{3})t}$

11. $6y''-5y'+y=0\quad y(0)=4,\ y'(0)=0$
$6r^2-5r+1=0$
$r=\frac{5\pm\sqrt{25-4(6)(1)}}{2(6)}=\frac{5\pm\sqrt{1}}{12}$
$r_1=\frac{1}{2},\ r_2=\frac{1}{3}$
$y=c_1e^{\frac{1}{2}t}+c_2e^{\frac{1}{3}t}$
$y'=\frac{1}{2}c_1e^{\frac{1}{2}t}+\frac{1}{3}c_2e^{\frac{1}{3}t}$

$y(0)=4=c_1+c_2$
$y'(0)=0=\frac{1}{2}c_1+\frac{1}{3}c_2$
$c_1=-\frac{2}{3}c_2$
$-\frac{2}{3}c_2+c_2=4\rightarrow c_2=12,\ c_1=-8$
$y=-8e^{\frac{1}{2}t}+12e^{\frac{1}{3}t}$

12. $y''+3y'=0 \quad y(0)=-2,\ y'(0)=3$

$r^2+3r=0$

$r(r+3)=0$

$r_1=-3 \quad r=0$

$y=c_1e^{-3t}+c_2$

$y'=-3c_1e^{-3t}$

$y(0)=-2=c_1+c_2$

$y'(0)=3=-3c_1 \rightarrow c_1=-1$

$c_2=-1$

$y=-e^{-3t}-1$

17. $y=c_1e^{2t}+c_2e^{-3t}$

$r_1=2,\ r_2=-3$

$(r-2)(r+3)=0$

$r^2+r-6=0$

$y''+y'-6y=0$

19. $y''-y=0 \quad y(0)=\frac{5}{4},\ y'(0)=-\frac{3}{4}$

$r^2-1=0 \quad r_1=1,\ r_2=-1$

$y=c_1e^t+c_2e^{-t}$

$y'=c_1e^t-c_2e^{-t}$

$y(0)=\frac{5}{4}=c_1+c_2$

$y'(0)=-\frac{3}{4}=c_1-c_2$

$\frac{1}{2}=2c_1 \rightarrow c_1=\frac{1}{4} \quad c_2=1$

$y=\frac{1}{4}e^t+e^{-t}$

21. $y''-y'-2y=0 \quad y(0)=\alpha,\ y'(0)=2$

$r^2-r-2=0$

$(r-2)(r+1)=0$

$y=c_1e^{2t}+c_2e^{-t}$

$y'=2c_1e^{2t}-c_2e^{-t}$

$y(0)=c_1+c_2=\alpha$

$y'(0)=2c_1-c_2=2$

$c_1=\alpha-c_2$

$2\alpha-2c_2-c_2=2$

$2\alpha-3c_2=2$

$\frac{2\alpha-2}{3}=c_2$

$c_1=\frac{3\alpha}{3}-\frac{2\alpha-2}{3}=\frac{\alpha+2}{3}$

$y=\left(\frac{\alpha+2}{3}\right)e^{2t}+\left(\frac{2\alpha-2}{3}\right)e^{-t}$

$$\lim_{t\to\infty}\left(\frac{\alpha+2}{3}\right)\underbrace{e^{2t}}_{\downarrow\ \infty}+\underbrace{\left(\frac{2\alpha-2}{3}\right)e^{-t}}_{\to 0}=0$$

So need $\frac{\alpha+2}{3}=0 \rightarrow \alpha=-2.$

23. $y'' - (2\alpha-1)y' + \alpha(\alpha-1)y = 0$

$r^2 - (2\alpha-1)r + \alpha(\alpha-1) = 0$

$(r-\alpha)(r-(\alpha-1)) = 0.$

$y = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$

$\lim_{t\to\infty} c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t} = 0$ need $\alpha, \alpha-1 < 0. \to \alpha < 0$

Then $\lim_{t\to\infty} c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t} = \pm\infty$ need $\alpha, \alpha-1 > 0 \to \alpha > 1.$

3-2 § 1-7, 10, 13, 17, 21, 23.

1. $e^{2t}, e^{\frac{-3t}{2}}$

$$W = \begin{vmatrix} e^{2t} & e^{\frac{-3t}{2}} \\ 2e^{2t} & -\frac{3}{2}e^{\frac{-3t}{2}} \end{vmatrix} = -\frac{3}{2}e^{\frac{1}{2}t} - 2e^{\frac{1}{2}t} = -\frac{7}{2}e^{\frac{1}{2}t} \neq 0$$

2. $\cos t, \sin t$

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

3. e^{-2t}, te^{-2t}

$$W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t} \neq 0.$$

4. x, xe^x

$$W(x, xe^x) = \begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = x^2e^x + xe^x - xe^x = x^2e^x$$

5. $e^t\sin t, e^t\cos t$

$$W(e^t\sin t, e^t\cos t) = \begin{vmatrix} e^t\sin t & e^t\cos t \\ e^t\sin t + e^t\cos t & e^t\cos t - e^t\sin t \end{vmatrix} = e^{2t}\sin t\cos t - e^{2t}\sin^2 t - e^{2t}\sin t\cos t - e^{2t}\cos^2 t$$

$$= -e^{2t}(\sin^2 t + \cos^2 t)$$

$$= -e^{2t}$$

6. $\cos^2\theta, 1+\cos 2\theta$

$$W(\cos^2\theta, 1+\cos 2\theta) = \begin{vmatrix} \cos^2\theta & \overset{2\cos^2\theta}{1+\cos 2\theta} \\ -2\sin\theta\cos\theta & \underset{-4\sin\theta\cos\theta}{-2\sin 2\theta} \end{vmatrix} = -4\sin\theta\cos^3\theta + 4\sin\theta\cos^3\theta = 0$$

7. $ty'' + 3y = t$, $y(1)=1$, $y'(1)=2$

$y'' + \frac{3y}{t} = 1$

$\frac{3}{t}$ cont where $t \neq 0$; 1 cont everywhere

$(-\infty,0)$, $(0,\infty)$

Then $t=1$ is in $(0,\infty)$

$(0,\infty)$

10. $y'' + (\cos t)y' + 3(\ln|t|)y = 0$ $\quad y(2)=3$, $y'(2)=1$

$\cos t$ cont. everywhere ; $\ln|t|$ cont where $t \neq 0$

$(-\infty,0)$, $(0,\infty)$

Then $t=2$ is in $(0,\infty)$

$(0,\infty)$

13. $t^2y'' - 2y = 0$ for $t>0$ $\quad y_1(t) = t^2$, $y_2(t) = t^{-1}$

$y_1'(t) = 2t$ $\quad$ so $t^2(2) - 2(t^2) = 0$

$y_1''(t) = 2$ $\quad$ y_1 is a sol

$y_2'(t) = -t^{-2}$ $\quad$ so $t^2(2t^{-3}) - 2(t^{-1}) = 2t^{-1} - 2t^{-1} = 0$

$y_2''(t) = 2t^{-3}$ $\quad$ y_2 is a sol.

$$W(y_1,y_2)(t_0) = \begin{vmatrix} t_0^2 & t_0^{-1} \\ 2t_0 & -t_0^{-2} \end{vmatrix} = -1-2 = -3 \neq 0$$

Then by Thm 3.2.4, $y = c_1y_1(t) + c_2y_2(t)$ for arbitrary c_1, c_2.

17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

$$W(f,g) = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = g'e^{2t} - 2ge^{2t} = 3e^{4t}$$

$g' - 2g = 3e^{2t}$ $\quad$ IF $e^{-\int 2dt} = e^{-2t}$

$e^{-2t}(g' - 2g) = e^{-2t}(3e^{2t}) = 3$

$\int \frac{d}{dt}[e^{-2t}g]\,dt = \int 3dt$

$e^{-2t}g = 3t + c$

$g(t) = 3te^{2t} + ce^{2t}$

21. $y'' + y' - 2y = 0,\ t_0 = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$y = c_1 e^{-2t} + c_2 e^t$. $\qquad y'(t) = -2c_1 e^{-2t} + c_2 e^t$

y_1 is a sol satisfying $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$

$\begin{cases} c_1 + c_2 = 1 \\ -2c_1 + c_2 = 0 \end{cases} \to c_2 = 2c_1 \qquad c_1 + 2c_1 = 1 \to c_1 = \frac{1}{3} \qquad c_2 = \frac{2}{3}$

$y_1 = \frac{1}{3} e^{-2t} + \frac{2}{3} e^t$

y_2 is a sol satisfying $\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$

$\begin{cases} c_1 + c_2 = 0 \to c_1 = -c_2 \\ -2c_1 + c_2 = 1 \qquad -2(-c_2) + c_2 = 1 \end{cases} \to c_2 = \frac{1}{3} \qquad c_1 = -\frac{1}{3}$

$y_2 = \frac{-1}{3} e^{-2t} + \frac{1}{3} e^t$

$W(y_1, y_2)(t_0) = W(y_1, y_2)(0) = \begin{vmatrix} \frac{1}{3} + \frac{2}{3} & \frac{-1}{3} + \frac{1}{3} \\ -\frac{2}{3} + \frac{2}{3} & \frac{2}{3} + \frac{1}{3} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

23. $y'' + 4y = 0 \qquad y_1(t) = \cos 2t,\ y_2(t) = \sin 2t.$

$y_1'(t) = -2\sin 2t \qquad$ so $-4\cos 2t + 4(\cos 2t) = 0$

$y_1''(t) = -4\cos 2t \qquad$ y_1 is a sol

$y_2'(t) = 2\cos 2t \qquad$ so $-4\sin 2t + 4(\sin 2t) = 0$

$y_2''(t) = -4\sin 2t \qquad$ y_2 is a sol

$W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2(1) = 2$

So y_1, y_2 form a FSS.