3-3 $\#$ 1-6, 8, 9, 15, 21

1. $f(t)=t^2+5t$, $g(t)=t^2-5t$

$K_1f+K_2g=0$

$K_1(t^2+5t)+K_2(t^2-5t)=0$

$(K_1+K_2)t^2+(5K_1-5K_2)t=0t^2+0t$

$\begin{cases} K_1+K_2=0 \\ 5K_1-5K_2=0 \rightarrow K_1=K_2 \end{cases}$ $K_1+K_1=0 \rightarrow K_1=0$, $K_2=0$ Linearly independent

2. $f(\theta)=\cos 3\theta$, $g(\theta)=4\cos^3\theta-3\cos\theta$.

Rewrite $\cos 3\theta=\cos(2\theta+\theta)$

$=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta.$

$=(2\cos^2\theta-1)\cos\theta-(2\sin\theta\cos\theta)\sin\theta$

$=2\cos^3\theta-\cos\theta-2\sin^2\theta\cos\theta$

$=2\cos^3\theta-\cos\theta(1+2\sin^2\theta)$

$=2\cos^3\theta-\cos\theta\left[1+2\left(\frac{1-\cos 2\theta}{2}\right)\right]$

$=2\cos^3\theta-\cos\theta[2-(2\cos^2\theta-1)]$

$=2\cos^3\theta-\cos\theta(3-2\cos^2\theta)$

$=4\cos^3\theta-3\cos\theta.$

$=g(\theta).$

So $K_1f+K_2g=0 \rightarrow K_1=1$, $K_2=-1$ Linearly dependent.

3. $f(t)=e^{\lambda t}\cos\mu t$ $g(t)=e^{\lambda t}\sin\mu t$ $\mu\neq 0$

$K_1e^{\lambda t}\cos\mu t+K_2e^{\lambda t}\sin\mu t=0e^{\lambda t}\cos\mu t+0e^{\lambda t}\sin\mu t \rightarrow K_1=K_2=0$

Linearly independent.

4. $f(x)=e^{3x}$, $g(x)=e^{3(x-1)}=e^{3x}e^{-3}$

$K_1f+K_2g=0$

$K_1e^{3x}+K_2e^{-3}e^{3x}=0 \rightarrow K_1=e^{-3}$, $K_2=-1$ Linearly dependent

5. $f(t) = 3t-5$, $g(t) = 9t-15$

$k_1(3t-5) + k_2(9t-15) = 0$

$(3k_1+9k_2)t + (-5k_1-15k_2) = 0t+0$

$$\begin{cases} 3k_1+9k_2 = 0 \\ -5k_1-15k_2 = 0 \end{cases} \quad \left[\begin{array}{cc|c} 3 & 9 & 0 \\ -5 & -15 & 0 \end{array}\right] \xrightarrow{2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 3 & 9 & 0 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow{R_1-3R_2 \to R_1}$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 3 & 0 \end{array}\right] \to \text{linearly dependent.}$$

6. $f(t) = t$, $g(t) = t^{-1}$

$k_1 t + k_2 t^{-1} = 0t + 0t^{-1} \to k_1 = k_2 = 0$. Linearly independent.

8. $f(x) = x^3$, $g(x) = |x|^3$

$k_1 x^3 + k_2 |x|^3 = 0$

If x is in $(0,b)$, then f,g are identical $[y_1 = 1 \cdot y_2]$

If x is in $(a,0)$, then f,g are negative of each other $[y_1 = -1 \cdot y_2]$

However if x is in (a,b), neither f,g is a fixed constant multiple of each other when 0 is in (a,b).

So if 0 is in (a,b), linear independent

otherwise linearly dependent.

9. $W(t) = t\sin^2 t$

Note there exist values of t such that $t\sin^2 t \neq 0$. Then the functions are linearly independent

15. $t^2 y'' - t(t+2)y' + (t+2)y = 0$

$$y'' - \left(\frac{t+2}{t}\right)y' + \left(\frac{t+2}{t^2}\right)y = 0$$

$$W(y_1,y_2)(t) = ce^{-\int p(t)dt} = ce^{\int \frac{t+2}{t}dt} = ce^{\int dt + \int \frac{2}{t}dt} = ce^{t+\ln t^2} = ct^2e^t$$

21. $t^2y'' - 2y' + (3+t)y = 0 \qquad W(y_1,y_2)(2) = 3$

$y'' - \left(\frac{2}{t^2}\right)y' + \left(\frac{3+t}{t^2}\right)y = 0$

$W(y_1,y_2)(t) = ce^{2\int t^{-2}dt} = ce^{-2t^{-1}}$

$W(y_1,y_2)(2) = ce^{-2(\frac{1}{2})} = ce^{-1} = 3 \rightarrow c = 3e$

$W(y_1,y_2)(4) = (3e)e^{-2(\frac{1}{4})} = 3e^{1-\frac{1}{2}} = 3e^{\frac{1}{2}} \approx 4.946.$

3-4 ~~8~~ 1, 4, 5, 7, 8, 11, 12, 16, 17, 19, 23

1. $e^{1+2i} = e^1e^{2i} = e^1(\cos 2 + i\sin 2)$

4. $e^{2-\frac{\pi}{2}i} = e^2e^{-\frac{\pi i}{2}} = e^2(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}) = -e^2 i$

5. $2^{1-i} = e^{\ln 2^{1-i}} = e^{(1-i)\ln 2} = e^{\ln 2}e^{-i\ln 2} = 2(\cos\ln 2 - i\sin\ln 2)$

7. $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0$

$r = \dfrac{2 \pm \sqrt{4 - 4(2)(1)}}{2} = \dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$y = c_1e^{(1+i)t} + c_2e^{(1-i)t}$

$= e^t(c_1\cos t + c_2\sin t)$

8. $y'' - 2y' + 6y = 0$

$r^2 - 2r + 6 = 0$

$r = \dfrac{2 \pm \sqrt{4 - 4(6)}}{2} = \dfrac{2 \pm \sqrt{-20}}{2} = \dfrac{2 \pm 2\sqrt{5}i}{2} = 1 \pm \sqrt{5}i$

$y = c_1e^{(1+\sqrt{5}i)t} + c_2e^{(1-\sqrt{5}i)t}$

$= e^t(c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t)$

11. $y''+6y'+13y=0$

$r^2+6r+13=0$

$r=\frac{-6\pm\sqrt{36-4(13)}}{2}=\frac{-6\pm\sqrt{-16}}{2}=\frac{-6\pm 4i}{2}=-3\pm 2i$

$y=c_1e^{(-3+2i)t}+c_2e^{(-3-2i)t}$

$=e^{-3t}(c_1\cos 2t+c_2\sin 2t)$

12. $4y''+9y=0$

$4r^2+9=0$

$r=\frac{0\pm\sqrt{0-4(9)(4)}}{2(4)}=\frac{\pm 12i}{8}=\pm\frac{3i}{2}$

$y=c_1e^{(0+\frac{3i}{2})t}+c_2e^{(0-\frac{3i}{2})t}$

$=c_1\cos\left(\frac{3t}{2}\right)+c_2\sin\left(\frac{3t}{2}\right)$

16. $y''+4y'+6.25y=0$

$r^2+4r+6.25=0$

$r=\frac{-4\pm\sqrt{16-4\left(\frac{25}{4}\right)(1)}}{2}=\frac{-4\pm\sqrt{-9}}{2}=-2\pm\frac{3i}{2}$

$y=c_1e^{(-2+\frac{3}{2}i)t}+c_2e^{(-2-\frac{3i}{2})t}$

$=e^{-2t}\left[c_1\cos\left(\frac{3t}{2}\right)+c_2\sin\left(\frac{3t}{2}\right)\right]$

17. $y''+4y=0 \quad y(0)=0,\ y'(0)=1$

$r^2+4=0$

$r=\frac{0\pm\sqrt{0-4(4)}}{2}=\frac{\pm 4i}{2}=\pm 2i$

$y=c_1e^{2it}+c_2e^{-2it}$

$=c_1\cos 2t+c_2\sin 2t$

$y'=-2c_1\sin 2t+2c_2\cos 2t$

$y(0)=0=c_1+0 \quad \to c_1=0$

$y'(0)=1=2c_2 \quad c_2=\frac{1}{2}$

$y(t)=\frac{1}{2}\sin 2t.$

19. $y''-2y'+5y=0 \quad y\left(\frac{\pi}{2}\right)=0,\ y'\left(\frac{\pi}{2}\right)=2$

$r^2-2r+5=0$

$r=\dfrac{2\pm\sqrt{4-4(5)}}{2}=\dfrac{2\pm\sqrt{-16}}{2}=1\pm 2i$

$y=c_1e^{(1+2i)t}+c_2e^{(1-2i)t}$

$\quad =e^t(c_1\cos 2t+c_2\sin 2t)$

$y\left(\frac{\pi}{2}\right)=e^{\pi/2}(c_1(-1)+0)=0 \quad \to c_1=0$

$y=e^t(c_2\sin 2t)$

$y'=e^t(c_2\sin 2t)+2c_2e^t\cos 2t=2$

$y'\left(\frac{\pi}{2}\right)=0+2c_2e^{\frac{\pi}{2}}(-1)=2 \quad \to c_2=e^{-\pi/2}$

$y=e^te^{-\frac{\pi}{2}}\sin 2t=e^{t-\pi/2}\sin 2t$

Then as t increases, y oscillates.

23. $3u''-u'+2u=0 \quad u(0)=2,\ u'(0)=0$

$3r^2-r+2=0$

$r=\dfrac{1\pm\sqrt{1-4(3)(2)}}{2(3)}=\dfrac{1\pm\sqrt{-23}}{6}=\dfrac{1\pm\sqrt{23}\,i}{6}$

$u=c_1e^{\left(\frac{1}{6}+\frac{\sqrt{23}}{6}i\right)t}+c_2e^{\left(\frac{1}{6}-\frac{\sqrt{23}}{6}i\right)t}$

$\quad =e^{\frac{1}{6}t}\left(c_1\cos\left(\frac{\sqrt{23}}{6}t\right)+c_2\sin\left(\frac{\sqrt{23}}{6}t\right)\right)$

$u(0)=2=c_1$

$u=e^{\frac{1}{6}t}\left(2\cos\left(\frac{\sqrt{23}}{6}t\right)+c_2\sin\left(\frac{\sqrt{23}}{6}t\right)\right)=2e^{\frac{1}{6}t}\cos\left(\frac{\sqrt{23}}{6}t\right)+c_2e^{\frac{1}{6}t}\sin\left(\frac{\sqrt{23}}{6}t\right)$

$u'=\frac{1}{3}e^{\frac{1}{6}t}\cos\left(\frac{\sqrt{23}}{6}t\right)-\frac{\sqrt{23}}{3}e^{\frac{1}{6}t}\sin\left(\frac{\sqrt{23}}{6}t\right)+\frac{c_2}{6}e^{\frac{1}{6}t}\sin\left(\frac{\sqrt{23}}{6}t\right)+\frac{\sqrt{23}}{6}c_2e^{\frac{1}{6}t}\cos\left(\frac{\sqrt{23}}{6}t\right)$

$u'(0)=\frac{1}{3}-0+0\ \frac{\sqrt{23}}{6}c_2=0 \quad \to c_2=\frac{-2}{\sqrt{23}}$

$u(t)=e^{\frac{1}{6}t}\left(2\cos\left(\frac{\sqrt{23}t}{6}\right)-\frac{2}{\sqrt{23}}\sin\left(\frac{\sqrt{23}t}{6}\right)\right)$

Now plot $|u(t)|$ and estimate t for $|u(t)|=10$