3-6 §1, 2, 5, 15

1. $y''-2y'-3y=3e^{2t}$

$y''-2y'-3y=0$

$r^2-2r-3=0$

$(r+1)(r-3)=0$

$y_h=c_1e^{-t}+c_2e^{3t}$

$y_p=Ae^{2t}$

$y_p'=2Ae^{2t}$

$y_p''=4Ae^{2t}$

$y_p''-2y_p'-3y_p=(4A-4A-3A)e^{2t}$

$=-3Ae^{2t}$

$=3e^{2t} \rightarrow A=-1$

$y=c_1e^{-t}+c_2e^{3t}-e^{2t}$

2. $y''+2y'+5y=3\sin 2t$

$y''+2y'+5y=0$

$r^2+2r+5=0$

$r=\dfrac{-2\pm\sqrt{4-4(5)}}{2}$

$=-1\pm 2i$

$y_h=e^{-t}(c_1\cos 2t+c_2\sin 2t)$

$y_p=A\sin 2t+B\cos 2t$

$y_p'=2A\cos 2t-2B\sin 2t$

$y_p''=-4A\sin 2t-4B\cos 2t.$

$y_p''+2y_p'+5y_p=(-4A-4B+5A)\sin 2t$

$+(-4B+4A+5B)\cos 2t$

$=(A-4B)\sin 2t+(4A+B)\cos 2t$

$=3\sin 2t$

$\begin{cases} A-4B=3 \\ 4A+B=0 \end{cases} \qquad A=\frac{3}{17} \qquad B=-\frac{12}{17}$

$y=e^{-t}(c_1\cos 2t+c_2\sin 2t)$

$+\frac{3}{17}\sin 2t-\frac{12}{17}\cos 2t.$

5. $y'' + 9y = t^2 e^{3t} + 6$

$y'' + 9y = 0$

$r^2 + 9 = 0$

$r = \pm 3i$

$y_h = c_1 \cos 3t + c_2 \sin 3t.$

$(D-3)^3[t^2 e^{3t}] = 0$

$D[6] = 0$

$D(D-3)^3(D^2+9)y = 0$

$$y = \underbrace{c_1 \cos 3t + c_2 \sin 3t}_{y_h} + \underbrace{c_3 e^{3t} + c_4 t e^{3t} + c_5 t^2 e^{3t} + c_6}_{y_p}$$

$y_p' = 3c_3 e^{3t} + c_4 e^{3t} + 3c_4 t e^{3t} + 2c_5 t e^{3t} + 3c_5 t^2 e^{3t}$

$y_p'' = 9c_3 e^{3t} + 3c_4 e^{3t} + 3c_4 e^{3t} + 9c_4 t e^{3t} + 2c_5 e^{3t} + 6c_5 t e^{3t} + 6_5 t e^{3t} + 9c_5 t^2 e^{3t}$

$\quad = 9c_3 e^{3t} + 6c_4 e^{3t} + 9c_4 t e^{3t} + 2c_5 e^{3t} + 12c_5 t e^{3t} + 9c_5 t^2 e^{3t}$

$y_p'' + 9y_p = (9c_5 + 9c_5)t^2 e^{3t} + (9c_4 + 12c_5 + 9c_4) t e^{3t} + (9c_3 + 6c_4 + 2_5 + 9c_3) e^{3t} + 9c_6$

$\quad = t^2 e^{3t} + 6$

$$\begin{cases} 18c_5 = 1 & c_5 = \frac{1}{18} \\ 18c_4 + 12c_5 = 0 & 18c_4 + \frac{2}{3} = 0 \rightarrow c_4 = -\frac{1}{27} \\ 18c_3 + 6c_4 + 2c_5 = 0 & 18c_3 - \frac{2}{9} + \frac{1}{9} = 0 \rightarrow c_3 = \frac{1}{162} \\ 9c_6 = 6 & c_6 = 2/3 \end{cases}$$

$$y = c_1 \cos 3t + c_2 \sin 3t + \frac{e^{3t}}{162} - \frac{te^{3t}}{27} + \frac{t^2 e^{3t}}{18} + \frac{2}{3}$$

$15\ y''-2y'+y=te^t+4 \qquad y(0)=1,\ y'(0)=1$

$y''-2y'+y=0$

$r^2-2r+1=0$

$(r-1)^2=0$

$y_h=c_1e^t+c_2te^t$

$$\begin{cases}(D-1)^2[te^t]=0\\ D[4]=0\end{cases}$$

$D(D-1)^2(D-1)^2y=0$

$$y=\underbrace{c_1e^t+c_2te^t}_{y_h}+\underbrace{c_3t^2e^t+c_4t^3e^t+c_5}_{y_p}$$

$y_p'=2c_3te^t+c_3t^2e^t+3c_4t^2e^t+c_4t^3e^t$

$y_p''=2c_3e^t+2c_3te^t+2c_3te^t+c_3t^2e^t+6c_4te^t+3c_4t^2e^t+3c_4t^2e^t+c_4t^3e^t$

$\quad=2c_3e^t+4c_3te^t+c_3t^2e^t+6c_4te^t+6c_4t^2e^t+c_4t^3e^t$

$y_p''-2y_p'+y_p=(c_4-2c_4+c_4)t^3e^t$

$\quad+(c_3+6c_4-2c_3-6c_4+c_3)t^2e^t$

$\quad+(4c_3+6c_4-4c_3)te^t$

$\quad+(2c_3)e^t$

$\quad+c_5$

$6c_4=1 \qquad c_4=\frac{1}{6}$

$2c_3=0 \qquad c_3=0$

$c_5=4 \qquad c_5=4$

$y=c_1e^t+c_2te^t+\frac{1}{6}t^3e^t+4.$

$y(0)=c_1+0+0+4=1 \qquad \rightarrow c_1=-3$

$y'=-3e^t+c_2e^t+c_2te^t+\frac{1}{2}t^2e^t+\frac{1}{6}t^3e^t$

$y'(0)=-3+c_2+0+0+0=1 \qquad \rightarrow c_2=4$

$y=-3e^t+4te^t+\frac{1}{6}t^3e^t+4$

3-7 §1,5.

1. $y''-5y'+6y=2e^t$

$y''-5y+6y=0$

$r^2-5r+6=0$

$(r-2)(r-3)=0$

$y_h=c_1\underset{\substack{\uparrow\\ y_1}}{e^{2t}}+c_2\underset{\substack{\uparrow\\ y_2}}{e^{3t}}$

$W(y_1,y_2)=\begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t}\end{vmatrix}=3e^{5t}-2e^{5t}=e^{5t}\neq 0$

$W_1=\begin{vmatrix} 0 & e^{3t} \\ 2e^{t} & 3e^{3t}\end{vmatrix}=-2e^{4t}$ $\qquad$ $W_2=\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & 2e^{t}\end{vmatrix}=2e^{3t}$

$u_1'=\dfrac{W_1}{W}=\dfrac{-2e^{4t}}{e^{5t}}=-2e^{-t}$ $\qquad$ $u_2'=\dfrac{W_2}{W}=\dfrac{2e^{3t}}{e^{5t}}=2e^{-2t}$,

$u_1=\int -2e^{-t}dt=2e^{-t}$ $\qquad$ $u_2=2\int e^{-2t}dt=-e^{-2t}$

$y_p=u_1y_1+u_2y_2$

$=(2e^{-t})(e^{2t})-(e^{-2t})(e^{3t})$

$=2e^t-e^t$

$=e^t$

5 $y'' + y = \tan t \quad . \quad 0 < t < \frac{\pi}{2}$

$$\left.\begin{aligned} &y''+y=0 \\ &r^2+1=0 \\ &r=\pm i \end{aligned}\right\} \quad y_h = c_1 \underset{\substack{\uparrow \\ y_1}}{\cos t} + c_2 \underset{\substack{\uparrow \\ y_2}}{\sin t}.$$

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t$$

$$W_1 = \begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix} = -\sin t \tan t \qquad W_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix} = \sin t.$$

$$u_1' = \frac{W_1}{W} = -\sin t \tan t \qquad u_2' = \frac{W_2}{W} = \sin t.$$

$$u_1 = -\int \sin t \tan t \, dt \qquad \int u\,dv = ux - \int v\,du$$

$u = \tan t \qquad dv = -\sin t$

$du = \sec^2 t \qquad v = \cos t$

$= \sin t - \int \sec t \, dt$

$= \sin t - \ln|\sec t + \tan t|$

$u_2 = \int \sin t \, dt = -\cos t.$

$y_p = u_1 y_1 + u_2 y_2$

$= (\sin t - \ln|\sec t + \tan t|)\cos t \; - \cos t \sin t$

$= \cos t\,[\sin t - \ln|\sec t + \tan t - \sin t]$

$= -\cos t\,[\ln|\sec t + \tan t|]$

$y = c_1 \cos t + c_2 \sin t - \cos t\,(\ln|\sec t + \tan t|)$

3-8 q 1,5,9.

1. $u = 3\cos 2t + 4\sin 2t$

$R = \sqrt{3^2+4^2} = 5$

$\omega_0 = 2$

$\delta = \tan^{-1}\left(\frac{4}{3}\right) \approx .927$

$u = 5\cos(2t - .927)$

Note these are equivalent.

$u(\frac{\pi}{2}) \approx -3$

Alt form

$\omega = 2$

$\varphi = \tan^{-1}\left(\frac{3}{4}\right) \approx .643$

$u = 5\sin(2t + .643)$

$u(\frac{\pi}{2}) \approx -3$

5. A mass weighing 2lb stretches 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine the position y of the mass at any time t. Find the frequency, period, and amplitude of the motion.

$m = \frac{W}{g} = \frac{2}{32} = \frac{1}{16}$ $s = 6\text{in}\left(\frac{1\text{ft}}{12\text{in}}\right) = \frac{1}{2}\text{ft}$

$mg = ks \Rightarrow 2 = k(\frac{1}{2}) \rightarrow k = 4$.

$my'' = -ky \Rightarrow \frac{1}{16}y'' = -4y$ subject to $\begin{cases} y(0) = \frac{3}{12} & \text{initial displacement} \\ y'(0) = 0 & \text{initial velocity} \end{cases}$

Standard form: $y'' + 64y = 0$

$r^2 + 64 = 0$

$r = \pm 8i$

$y = c_1\cos 8t + c_2\sin 8t$

$y(0) = c_1 = \frac{3}{12} = \frac{1}{4}$

$y = \frac{1}{4}\cos 8t + c_2\sin 8t$

$y' = -2\sin 2t + 8c_2\cos 8t$

$y'(0) = 0 + 8c_2 = 0 \quad \rightarrow c_2 = 0$

$y = \frac{1}{4}\cos 8t$

$A=\sqrt{c_1^2+c_2^2}=\sqrt{\left(\frac{1}{4}\right)^2+0}=\frac{1}{4}$.

$\omega=8$

frequency: $\frac{1}{T}=\frac{\omega}{2\pi}=\frac{8}{2\pi}=\frac{4}{\pi}$

period: $T=\frac{2\pi}{\omega}=\frac{2\pi}{8}=\frac{\pi}{4}$

9 A mass of 20g stretches a spring 5cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 $\frac{\text{dyne-s}}{\text{cm}}$. If the mass is pulled down an additional 2cm and then released, find its position y at any time t. Determine the quasi period and the quasi frequency. Determine the ratio of the quasi period to the period of the corresponding undamped motion.

$W=mg=20\left(\frac{9.8m}{s^2}\right)\left(\frac{100cm}{1m}\right)=19600$

$19600=5K \rightarrow K=3920.$

$my''=-Ky-\beta y' \Rightarrow 20y''=-3920y-400y'$

Standard form $y''+20y'+196y=0$ $\begin{cases} y(0)=2 & \text{initial displacement} \\ y'(0)=0 & \text{initial velocity} \end{cases}$

$r^2+20r+196=0$

$r=\frac{-20\pm\sqrt{400-4(196)}}{2}=\frac{-20\pm\sqrt{-384}}{2}=\frac{-20\pm 8\sqrt{6}i}{2}=-10\pm 4\sqrt{6}i$

$y=e^{-10t}(c_1\cos 4\sqrt{6}t+c_2\sin 4\sqrt{6}t)$

$y(0)=1(c_1+0)=2 \rightarrow c_1=2$

$y=e^{-10t}(2\cos 4\sqrt{6}+c_2\sin 4\sqrt{6}t)$

$y'=-10e^{-10t}(2\cos 4\sqrt{6}t+c_2\sin 4\sqrt{6}t)+e^{-10t}(-8\sqrt{6}\sin 4\sqrt{6}+4\sqrt{6}c_2\cos 4\sqrt{6}t)$

$y'(0)=-10(2+0)+1(0+4\sqrt{6}c_2)=0 \rightarrow c_4=\frac{5}{\sqrt{6}}$

$y=e^{-10t}\left(2\cos 4\sqrt{6}t+\frac{5}{\sqrt{6}}\sin 4\sqrt{6}t\right)$

Quasi frequency: $4\sqrt{6}$

Quasi period: $\frac{2\pi}{\sqrt{w^2-\lambda^2}} = \frac{2\pi}{4\sqrt{6}} = \frac{\pi}{2\sqrt{6}}$

undamped period: $T = \frac{2\pi}{w} = 2\pi\left(\frac{m}{K}\right)^{1/2} = 2\pi\left(\frac{20}{3920}\right)^{1/2} = \frac{\pi}{7}$

$\frac{\text{Quasi period}}{\text{undamped period}} = \frac{\pi/2\sqrt{6}}{\pi/7} = \frac{7}{2\sqrt{6}}$

4-1 # 3, 7, 11.

3. $t(t-1)y^{iv} + e^t y'' + 4t^2 y = 0$

$y^{iv} + \frac{e^t}{t(t-1)} y'' + \frac{4t}{t-1} y = 0$

cont when $t \neq 0, 1$ cont when $t \neq 1$

Intervals: $(-\infty, 0)$, $(0, 1)$, + $(1, \infty)$.

7. $f_1(t) = 2t-3$ $\quad f_2(t) = t^2+1$ $\quad f_3(t) = 2t^2 - t$

Note since these functions are polynomials, the implied interval is $(-\infty, \infty)$.

$$W(f_1, f_2, f_3)(0) = \begin{vmatrix} 2(0)-3 & 0^2+1 & 2(0)^2-0 \\ 2(0) & 2(0) & 4(0)-1 \\ 2 & 2 & 4 \end{vmatrix} = \begin{vmatrix} -3 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} -3 & 1 \\ 2 & 2 \end{vmatrix} = -8$$

linearly independent.

11. $y''' + y' = 0$ $\quad y_1 = 1,\ y_2 = \cos t,\ y_2 = \sin t.$

$$W(1, \cos t, \sin t)(0) = \begin{vmatrix} 1 & \cos(0) & \sin(0) \\ 0 & -\sin(0) & \cos(0) \\ 0 & -\cos(0) & -\sin(0) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

4-2 g 1, 11, 15

1. $1+i$

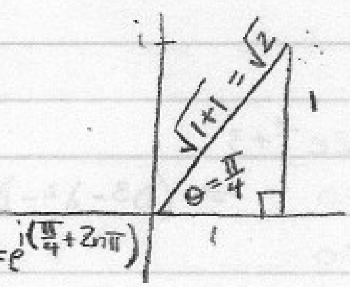
$R = \sqrt{2}$

$\theta = \frac{\pi}{4}$

$1+i = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\frac{\pi}{4}}$

$\cos\left(\frac{\pi}{4}+2n\pi\right) + i\sin\left(\frac{\pi}{4}+2n\pi\right) = e^{i\left(\frac{\pi}{4}+2n\pi\right)}$

$\Rightarrow Re^{i\theta} = \sqrt{2}\, e^{i\left(\frac{\pi}{4}+2n\pi\right)}$



11. $y''' - y'' - y' + y = 0$

$r^3 - r^2 - r + 1 = 0$

$r^2(r-1) - (r-1) = 0$

$(r-1)(r^2-1) = 0$

$(r-1)(r-1)(r+1) = 0$

$y = c_1e^t + c_2te^t + c_3e^{-t}$

15. $y^{vi} + y = 0$

$r^6 + 1 = 0$

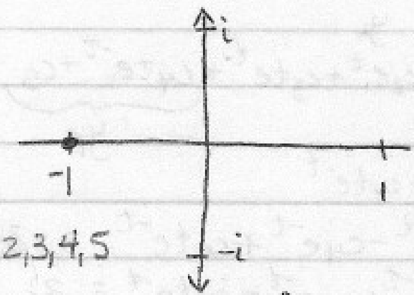
$r^6 = -1$

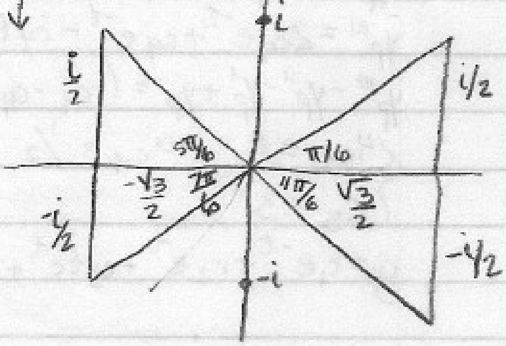
$\Rightarrow (-1)^{1/6} = e^{i(\pi+2n\pi)/6}$ $\quad n = 0, 1, 2, 3, 4, 5$

$e^{\frac{i\pi}{6}} = \frac{\sqrt{3}+i}{2}$

$e^{i(\pi+2\pi)/6} = e^{i\pi/2} = i$

$e^{i(\pi+4\pi)/6} = e^{i5\pi/6} = \frac{-\sqrt{3}+i}{2}$





$$e^{i(\pi+6\pi)/6} = e^{\frac{i7\pi}{6}} = \frac{-\sqrt{3}-i}{2}$$

$$e^{i(\pi+8\pi)/6} = e^{i\frac{3\pi}{2}} = -i$$

$$e^{i(\pi+10\pi)/6} = e^{i11\pi/6} = \frac{\sqrt{3}-i}{2}$$

$$y = e^{\frac{\sqrt{3}}{2}t}\left(c_1\cos\left(\tfrac{t}{2}\right) + c_2\sin\left(\tfrac{t}{2}\right)\right) + e^{-\frac{\sqrt{3}}{2}t}\left(c_3\cos\left(\tfrac{t}{2}\right) + c_4\sin\left(\tfrac{t}{2}\right)\right) + c_5\cos t + c_6\sin t.$$

4-3 #1, 13

1. $y''' - y'' - y' + y = 2e^{-t} + 3$

$y''' - y'' - y' + y = 0 \quad \Rightarrow (D^3 - D^2 - D + 1)y = 0$

$r^3 - r^2 - r + 1 = 0$

$r^2(r-1) - (r-1) = 0$

$(r^2-1)(r-1) = 0$

$(r+1)(r-1)(r-1) = 0$

$y_h = c_1e^{-t} + c_2e^{t} + c_3te^{t}.$

$(D+1)[2e^{-t}] = 0$

$D[3] = 0$

$D(D+1)(D^3 - D^2 - D + 1)y = D(D+1)[2e^{-t}+3] = 0$

$\underbrace{D(D+1)}_{y_p}\underbrace{(D-1)^2(D+1)}_{y_h}y = 0$

$y = c_1e^{-t} + c_2e^{t} + c_3te^{t} + \underbrace{c_4te^{-t} + c_5}_{y_p}$

$y_p' = c_4e^{-t} - c_4te^{-t}$

$y_p'' = -c_4e^{-t} - c_4e^{-t} + c_4te^{-t} = -2c_4e^{-t} + c_4te^{-t}$

$y_p''' = 2c_4e^{-t} + c_4e^{-t} - c_4te^{-t} = 3c_4e^{-t} - c_4te^{-t}$

$y_p''' - y_p'' - y_p' + y_p = (-c_4 - c_4 + c_4 + c_4)te^{-t} + (3c_4 + 2c_4 - c_4)e^{-t} + c_5$

$\begin{cases} 4c_4 = 2 \rightarrow c_4 = 1/2 \\ c_5 = 3 \end{cases}$

$y = c_1e^{-t} + c_2e^{t} + c_3te^{t} + \frac{1}{2}te^{-t} + 3$

13. $y''' - 2y'' + y' = t^3 + 2e^t$

$y''' - 2y'' + y' = 0 \Rightarrow (D^3 - 2D^2 + D)y = D(D-1)^2 y = 0$

$r^3 - 2r^2 + r = 0$

$r(r^2 - 2r + 1) = 0$

$r(r-1)^2 = 0$

$Y_h = c_1 + c_2 e^t + c_3 t e^t$

$D^4[t^3] = 0$

$(D-1)[2e^t] = 0$

$\underbrace{D^4(D-1)}_{y_p} \underbrace{D(D-1)^2}_{y_h} y = D^4(D-1)[t^3 + 2e^t] = 0$

$y = \underbrace{c_1 e^t + c_2 t e^t + c_3}_{y_h} + \underbrace{c_4 t^2 e^t + c_5 t + c_6 t^2 + c_7 t^3 + c_8 t^4}_{y_p}$

3-9 like 1, 5

2. $\sin 7t - \sin 6t$

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v.$

$\sin(u+v) - \sin(u-v) = 2\cos u \sin v.$

Let $u + v = 7t$ $\qquad u = \frac{13t}{2} \qquad v = \frac{t}{2}$

$\underline{u - v = 6t}$

$2u = 13t$

Then $\sin 7t - \sin 6t = 2\cos\left(\frac{13t}{2}\right)\sin\left(\frac{t}{2}\right)$

6 A mass of 5Kg stretches a spring 10cm. The mass is acted on by an external force of $10\sin(\frac{t}{2})$N and moves in a medium that imparts a viscous force of 2N when the speed of the mass is 4cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3cm/s, formulate the IVP describing the motion of the mass.

$W = mg = 5(9.8) = 49$

$s = 10\text{cm} \to .1\text{m}$

$.1K = 49 \to K = 490$

$$\beta = \frac{2N}{\left(\frac{4\text{cm}}{s}\right)\left(\frac{1\text{m}}{100\text{cm}}\right)} = 50$$

$my'' + \beta y' + Ky = f(t) \Rightarrow 5y'' + 50y' + 490y = 10\sin(\frac{t}{2})$

$y'' + 10y' + 98y = 2\sin(t/2)$

subject to $\begin{cases} y(0) = 0 \\ y'(0) = \frac{3\text{cm}}{s} \Rightarrow .03 \end{cases}$