

5-5 g 1, 12, 14

$$1. x^2 y'' + 4xy' + 2y = 0, x \neq 0$$

$$\text{Let } y = x^r$$

$$y' = r x^{r-1}$$

$$y'' = r(r-1) x^{r-2}$$

$$x^2 [r(r-1) x^{r-2}] + 4x [r x^{r-1}] + 2x^r = 0$$

$$r(r-1)x^r + 4rx^r + 2x^r = 0$$

$$r^2 - r + 4r + 2 = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$y(x) = c_1 x^{-1} + c_2 x^{-2}$$

$$12. x^2 y'' - 4xy' + 4y = 0, x \neq 0$$

$$x^2 [r(r-1) x^{r-2}] - 4x [r x^{r-1}] + 4x^r = 0$$

$$(r^2 - r) x^r - 4r x^r + 4x^r = 0$$

$$r^2 - r - 4r + 4 = 0$$

$$r^2 - 5r + 4 = 0$$

$$(r-1)(r-4) = 0$$

$$y(x) = c_1 x + c_2 x^4$$

$$14. 4x^2 y'' + 8xy' + 17y = 0 \quad \begin{cases} y(1) = 2 \\ y'(1) = -3 \end{cases}$$

$$4x^2 [(r^2 - r) x^{r-2}] + 8x [r x^{r-1}] + 17x^r = 0$$

$$4(r^2 - r)x^r + 8rx^r + 17x^r = 0$$

$$4r^2 - 4r + 8r + 17 = 0$$

$$4r^2 + 4r + 17 = 0$$

$$r^2 + r + \frac{17}{4} = 0$$

$$r = \frac{-1 \pm \sqrt{1^2 - 4\left(\frac{17}{4}\right)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-16}}{2}$$

$$= -\frac{1}{2} \pm 2i$$

$$y(x) = x^{-1/2} [c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)]$$

$$y(1) = 1 [c_1(1) + c_2(0)] = 2 \Rightarrow c_1 = 2$$

$$y(x) = x^{-1/2} [2 \cos(2 \ln x) + c_2 \sin(2 \ln x)]$$

$$y'(x) = -\frac{1}{2} x^{-3/2} [2 \cos(2 \ln x) + c_2 \sin(2 \ln x)] + x^{-1/2} \left[-\frac{4}{x} \sin(2 \ln x) + \frac{2c_2}{x} \cos(2 \ln x) \right]$$

$$y'(1) = -\frac{1}{2} [2(1) + c_2(0)] + 1 [0 + 2c_2(1)] = -3$$

$$\Rightarrow 2c_2 - 1 = -3$$

$$\Rightarrow 2c_2 = -2$$

$$\Rightarrow c_2 = -1$$

$$y(x) = x^{-1/2} [2 \cos(2 \ln x) - \sin(2 \ln x)]$$

5-58 3, 16 (like 1, 12, 14)

3. $x^2 y'' - 3xy' + 4y = 0$, $x > 0$

$$y = x^r$$

$$y' = r x^{r-1}$$

$$y'' = r(r-1)x^{r-2}$$

$$x^2 [r(r-1)x^{r-2}] - 3x [r x^{r-1}] + 4x^r = 0$$

$$(r^2 - r)x^r - 3rx^r + 4x^r = 0$$

$$x^r [r^2 - r - 3r + 4] = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0 \quad r_1 = 2$$

$$y_1 = x^{r_1} = x^2$$

$$y_2 = x^{r_1} \ln|x| = x^2 \ln|x|$$

$$\text{So } y = c_1 x^2 + c_2 x^2 \ln|x|$$

← For future reference

the characteristic eq

for $ax^2 y'' + bxy' + cy = 0$

is $ar^2 + (b-a)r + c = 0$

16. $x^2 y'' + 3xy' + 5y = 0$ $y(1) = 1$, $y'(1) = -1$

$$a=1 \quad b=3 \quad c=5$$

$$r^2 + (3-1)r + 5 = 0$$

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} \Rightarrow \frac{-2 \pm 4i}{2} \Rightarrow -1 \pm 2i$$

$$y = x^{-1} (c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x))$$

$$y(1) = 1(c_1 + 0) = 1 \rightarrow c_1 = 1$$

$$y' = -x^{-2} (\cos(2 \ln x) + c_2 \sin(2 \ln x)) + x^{-1} \left(\frac{-2}{x} \sin(2 \ln x) + \frac{2c_2}{x} \cos(2 \ln x) \right)$$

$$y'(1) = -1(1 + 0) + 1(0 + 2c_2) = -1 \rightarrow c_2 = 0$$

$$\text{So } y = x^{-1} \cos(2 \ln x)$$