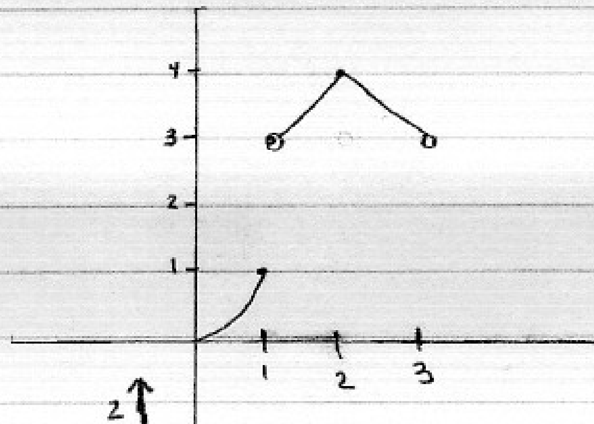
6-1 # 1, 3, 6

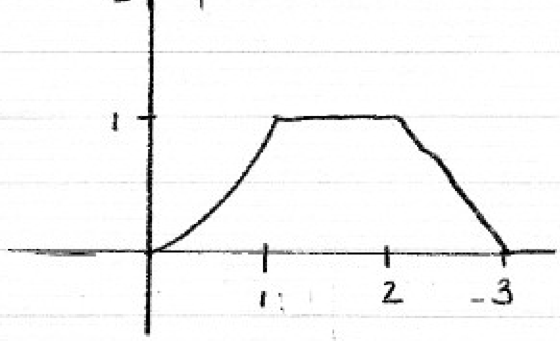
Determine whether f is continuous, piecewise continuous, or neither on (0, 3).

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ t+2, & 1 < t \le 2 \\ 6-t, & 2 < t \le 3 \end{cases}$



piecewise continuous.

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$



continuous

6. $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$

$u = \cos at \quad du = -a\sin at$

$dv = e^{-st} \quad v = -\frac{1}{s}e^{-st}$

$$= -\frac{e^{-st}\cos at}{s}\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt$$

$$= \frac{1}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt \qquad , s>0$$

$u = \sin at \quad du = a\cos at$

$dv = e^{-st} \quad v = -\frac{1}{s}e^{-st}$

$$= \frac{1}{s} - \frac{a}{s}\left[\frac{-e^{-st}\sin at}{s}\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at \, dt \right]$$

$$= \frac{1}{s} - \frac{a}{s}\left[0 + \frac{a}{s}\mathcal{L}\{\cos at\} \right]$$

Then $\mathcal{L}\{\cos at\} + \frac{a^2}{s^2}\mathcal{L}\{\cos at\} = \frac{1}{s}$

$$\left(1 + \frac{a^2}{s^2}\right)\mathcal{L}\{\cos at\} = \left(\frac{s^2+a^2}{s^2}\right)\mathcal{L}\{\cos at\} = \frac{1}{s}$$

So $\mathcal{L}\{\cos at\} = \frac{s}{s^2-a^2}$, $s>0$.

6-2 #1,3,11

1. $\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = 3\mathcal{L}^{-1}\left\{\left(\frac{2}{2}\right)\left(\frac{1}{s^2+2^2}\right)\right\}$

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{3}{2}\sin 2t.$$

3. $\mathcal{L}^{-1}\left\{\frac{2}{s^2+3s-4}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+4)(s-1)}\right\}$

Partial Fractions: $\frac{2}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1} = \frac{A(s-1)+B(s+4)}{(s+4)(s-1)}$

$s=1$ $\quad 5B=2 \rightarrow B = 2/5$

$s=-4$ $\quad -5A=2 \rightarrow A = -2/5$

$$= \mathcal{L}^{-1}\left\{ \frac{-\frac{2}{5}}{s+4} + \frac{\frac{2}{5}}{s-1} \right\}$$

$$= \frac{-2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{-2}{5}e^{-4t} + \frac{2}{5}e^{t}$$

11. $y'' - y' - 6y = 0 \qquad y(0) = 1,\ y'(0) = -1$

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\{y\} = 0$$

$$(s^2 Y(s) - sy(0) - y'(0)) - (sY(s) - y(0)) - 6Y(s) = 0$$

$$(s^2 - s - 6)Y(s) - sy(0) - y'(0) + y(0) = 0$$

$$(s^2 - s - 6)Y(s) = s - 2$$

$$Y(s) = \frac{s-2}{s^2 - s - 6} = \frac{s-2}{(s-3)(s+2)}$$

Partial Fractions: $\frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{A(s+2) + B(s-3)}{(s-3)(s+2)}$

$s = -2 \qquad -5B = -4 \rightarrow B = \frac{4}{5}$

$s = 3 \qquad 5A = 1 \rightarrow A = \frac{1}{5}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{5}}{s-3} + \frac{\frac{4}{5}}{s+2} \right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

6-3 § 11, 14, 15

11. $f(t) = (t-3)\mathcal{U}(t-2) - (t-2)\mathcal{U}(t-3)$

$= (t-2)\mathcal{U}(t-2) - \mathcal{U}(t-2) - (t-3)\mathcal{U}(t-3) - \mathcal{U}(t-3)$

$\mathcal{L}\{(t-2)\mathcal{U}(t-2) - \mathcal{U}(t-2) - (t-3)\mathcal{U}(t-3) - \mathcal{U}(t-3)\}$

$= \mathcal{L}\{(t-2)\mathcal{U}(t-2)\} - \mathcal{L}\{\mathcal{U}(t-2)\} - \mathcal{L}\{(t-3)\mathcal{U}(t-3)\} - \mathcal{L}\{\mathcal{U}(t-3)\}$.

$= e^{-2s}\mathcal{L}\{t\} - \frac{e^{-2s}}{s} - e^{-3s}\mathcal{L}\{t\} - \frac{e^{-3s}}{s}$

$= \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}$.

14. $F(s) = \frac{e^{-2s}}{s^2+s-2}$

$\mathcal{L}^{-1}\left\{e^{-2s}\underbrace{\left(\frac{1}{s^2+s-2}\right)}_{F(s)}\right\}$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s-1)}\right\}$

$\frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$

$s=1 \quad 3B = 1 \rightarrow B = 1/3$

$s=-2 \quad -3A = 1 \rightarrow A = -1/3$

$\mathcal{L}^{-1}\left\{\frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1}\right\} = \frac{-1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$= \frac{-1}{3}e^{-2t} + \frac{1}{3}e^{t}$

$= f(t)$

Recall: $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$

$a=2$

$$f(t-2) = -\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{t-2}$$

$$f(t-2)\,\mathcal{U}(t-2) = \frac{1}{3}\left(e^{t-2} - e^{-2(t-2)}\right)\mathcal{U}(t-2)$$

15 $G(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$.

$$\mathcal{L}^{-1}\left\{e^{-2s}\underbrace{\left(\frac{2(s-1)}{s^2-2s+2}\right)}_{F(s)}\right\}$$

From table: $\mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$ $\quad s>a$

$\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$ $\quad s>a$.

$(s-a)^2+b^2 = s^2-2as+(a^2+b^2) = s^2-2s+2$

$a=1 \quad a^2=1$

$b=1 \quad b^2=1$

$$\mathcal{L}^{-1}\left\{\frac{2(s-1)}{s^2-2s+2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1^2}\right\}$$

$$= 2e^t\cos t$$

$$= f(t)$$

$a=2$

$$f(t-2)\,\mathcal{U}(t-2) = 2e^{t-2}\cos(t-2)\,\mathcal{U}(t-2)$$

6-4 q 1,5,9.

1. $y''+y=f(t)$ $\begin{cases} y(0)=0 \\ y'(0)=1 \end{cases}$ $f(t)=\begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & t \ge \frac{\pi}{2} \end{cases}$

$= 1-\mathcal{U}(t-\pi/2)$

$\mathcal{L}\{y''+y\}=\mathcal{L}\{1-\mathcal{U}(t-\frac{\pi}{2})\}$

$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{1\}-\mathcal{L}\{\mathcal{U}(t-\frac{\pi}{2})\}$

$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{1}{s}-\frac{e^{-\frac{\pi}{2}s}}{s}$

$(s^2+1)Y(s)-1=\frac{1}{s}-\frac{e^{-\frac{\pi}{2}s}}{s}$

$(s^2+1)Y(s)=\frac{1}{s}-\frac{e^{-\frac{\pi}{2}s}}{s}+1$

$Y(s)=\frac{1}{s(s^2+1)}-e^{-\frac{\pi}{2}s}\underbrace{\left(\frac{1}{s(s^2+1)}\right)}_{F(s)}+\frac{1}{s^2+1}$

$F(s)=\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}=\frac{A(s^2+1)+Bs^2+Cs}{s(s^2+1)}$

$s=0$ $A=1$

$B=-1$

$C=0$

$f(t)=\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s}{s^2+1}\right\}$

$=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)}\right\}$

$=1-\cos t$

Then $\mathcal{L}^{-1}\{e^{-2s}F(s)\}=f(t-\frac{\pi}{2})\mathcal{U}(t-\frac{\pi}{2})$

Then $y(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-\frac{\pi}{2}s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{s^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= 1 - \cos t - f(t-\frac{\pi}{2})\mathcal{U}(t-\frac{\pi}{2}) + \sin t$

$= 1 - \cos t - (1-\cos(t-\frac{\pi}{2}))\mathcal{U}(t-\frac{\pi}{2}) + \sin t$

$= 1 - \cos t - (1-\sin t)\mathcal{U}(t-\frac{\pi}{2}) + \sin t.$

5. $y'' + 3y' + 2y = f(t)$ $\quad \begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$ $\quad f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$

$= 1 - \mathcal{U}(t-10)$

$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{1 - \mathcal{U}(t-10)\}$

$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{\mathcal{U}(t-10)\}$

$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$

$(s^2+3s+2)Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$

$Y(s) = \frac{1}{s(s^2+3s+2)} - e^{-10s}\Big(\underbrace{\frac{1}{s(s^2+3s+2)}}_{F(s)}\Big)$

Partial Fractions:

$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{A(s+1)(s+2) + Bs(s+2) + Cs(s+1)}{s(s+1)(s+2)}$

$s=0 \quad 2A = 1 \rightarrow A = \frac{1}{2}$

$s=-1 \quad -B = 1 \rightarrow B = -1$

$s=-2 \quad 2C = 1 \rightarrow C = \frac{1}{2}$

Then $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Then $\mathcal{L}^{-1}\{e^{-10s}F(s)\} = f(t-10)\,\mathcal{U}(t-10)$

So $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{\left(\frac{1/2}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right) - e^{-10s}\left(\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/2}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right\} - \mathcal{L}^{-1}\left\{e^{-10s}\left(\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)\right\}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} + f(t-10)\,\mathcal{U}(t-10)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} + \left(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right)\mathcal{U}(t-10).$$

9. $y'' + y = g(t)$ $\quad \begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$ $\quad g(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3 & t \ge 6 \end{cases}$

$$= \frac{t}{2} - \frac{t}{2}\mathcal{U}(t-6) + 3\,\mathcal{U}(t-6)$$

$$= \frac{t}{2} - \frac{1}{2}\left[t\,\mathcal{U}(t-6) - 6\,\mathcal{U}(t-6)\right]$$

$$= \frac{t}{2} - \frac{1}{2}\left[(t-6)\,\mathcal{U}(t-6)\right].$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{t/2 - \tfrac{1}{2}(t-6)\,\mathcal{U}(t-6)\right\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \tfrac{1}{2}\mathcal{L}\{t\} - \tfrac{1}{2}\mathcal{L}\{(t-6)\,\mathcal{U}(t-6)\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{2s^2} - \frac{1}{2}e^{-6s}\mathcal{L}\{t\}$$

$$(s^2+1)Y(s) - 0 - 1 = \frac{1}{2s^2} - \frac{1}{2s^2}e^{-6s} + 1$$

$$Y(s) = \frac{\frac{1}{2}}{s^2(s^2+1)} - e^{-6s}\left(\frac{\frac{1}{2}}{s^2(s^2+1)}\right) + \frac{1}{s^2+1} \qquad \text{Let } F(s) = \frac{\frac{1}{2}}{s^2(s^2+1)}$$

Partial Fractions: $\frac{\frac{1}{2}}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+1}$

$$\frac{1}{2} = As(s^2+1) + B(s^2+1) + Cs^3 + Ds^2$$

$s=0 \quad \frac{1}{2} = B$

$s=i \quad \frac{1}{2} = -Ci - D \quad \Rightarrow C=0,\ D=-\frac{1}{2}$

$s=1 \quad \frac{1}{2} = 2A + 2(\frac{1}{2}) + 1(0) + 1(-\frac{1}{2}) \quad \Rightarrow A=0$

Then $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+1}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}t - \frac{1}{2}\sin(t)$$

Recall $\mathcal{L}^{-1}\{e^{-6s}F(s)\} = f(t-6)\,\mathcal{U}(t-6)$

So $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+1} - e^{-6s}\left(\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+1}\right) + \frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{s^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}t - \frac{1}{2}\sin(t) - \frac{1}{2}(t-6)\,\mathcal{U}(t-6) + \frac{1}{2}\sin(t-6)\,\mathcal{U}(t-6) + \sin(t)$$

$$= \frac{1}{2}t + \frac{1}{2}\sin(t) + \frac{1}{2}\left[\sin(t-6) - (t-6)\right]\mathcal{U}(t-6)$$