6.5 §1,9.

1. $y''+2y'+2y=\delta(t-\pi) \qquad y(0)=1,\ y'(0)=0$

$\mathcal{L}\{y''+2y'+2y\}=\mathcal{L}\{\delta(t-\pi)\}$

$\mathcal{L}\{y''\}+2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\pi)\}$

$(s^2Y(s)-sy(0)-y'(0))+2(sY(s)-y(0))+2Y(s)=e^{-\pi s}$

$(s^2+2s+2)Y(s)-s-2=e^{-\pi s}$

$(s^2+2s+2)Y(s)=e^{-\pi s}+s+2$

$$Y(s)=\frac{e^{-\pi s}}{s^2+2s+2}+\frac{s+2}{s^2+2s+2}$$

$$G(s)=\frac{1}{s^2+2s+2}=\frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=e^{-t}\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1+1}{(s+1)^2+1}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=e^{-t}\cos(t)+e^{-t}\sin(t)$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}+\frac{s+2}{(s+1)^2+1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^2+1}\right\}$$

$$=e^{-(t-\pi)}\sin(t-\pi)\,\mathcal{U}(t-\pi)+e^{-t}\cos(t)+e^{-t}\sin(t)$$

$$=-e^{-(t-\pi)}\sin(t)\,\mathcal{U}((t-\pi)+e^{-t}\cos(t)+e^{-t}\sin(t)$$

9. $y''+y = \mathcal{U}(t-\frac{\pi}{2}) + 3\delta(t-\frac{3\pi}{2}) - \mathcal{U}(t-2\pi)$ $\qquad y(0)=y'(0)=0.$

$\mathcal{L}\{y''+y\} = \mathcal{L}\{\mathcal{U}(t-\frac{\pi}{2}) + 3\delta(t-\frac{3\pi}{2}) - \mathcal{U}(t-2\pi)\}$

$\mathcal{L}\{y''\}+\mathcal{L}\{y\}+\mathcal{L}\{\mathcal{U}(t-\frac{\pi}{2})\}+3\mathcal{L}\{\delta(t-\frac{3\pi}{2})\}-\mathcal{L}\{\mathcal{U}(t-2\pi\}$

$s^2Y(s) - s\underset{0}{\underset{\Downarrow}{y(0)}} - \underset{0}{\underset{\Downarrow}{y'(0)}} + Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s} + 3e^{-\frac{3\pi}{2}s} - \frac{e^{-2\pi s}}{s}$

$(s^2+1)Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s} + 3e^{-\frac{3\pi}{2}s} - \frac{e^{-2\pi s}}{s}$

$Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s(s^2+1)} + \frac{3e^{-\frac{3\pi}{2}s}}{s^2+1} - \frac{e^{-2\pi s}}{s(s^2+1)}$

$G(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$

$1 = A(s^2+1) + s(Bs+C)$

$s=0,\ A=1.$

$s=i,\ 1=-B+iC \implies B=-1$

$C=0.$

$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = 1-\cos(t)$

$H(s) = \frac{1}{s^2+1} \implies \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t).$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{s(s^2+1)} + \frac{3e^{-\frac{3\pi}{2}s}}{s^2+1} - \frac{e^{-2\pi s}}{s(s^2+1)}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{s(s^2+1)}\right\} + 3\mathcal{L}^{-1}\left\{\frac{e^{\frac{-3\pi s}{2}}}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s(s^2+1)}\right\}$

$= (1-\cos(t-\frac{\pi}{2}))\mathcal{U}(t-\frac{\pi}{2}) + 3\sin(t-\frac{3\pi}{2})\mathcal{U}(t-\frac{3\pi}{2}) - (1-\cos(t-2\pi))\mathcal{U}(t-2\pi)$

$= (1-\sin(t))\,\mathcal{U}(t-\frac{\pi}{2}) + 3\cos(t)\,\mathcal{U}(t-\frac{3\pi}{2}) - (1-\cos(t))\mathcal{U}(t-2\pi)$