MTH 204

Fall 2005

Exam 1

Math 204

Sections A & C (Wintz)

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Fall 2005

Name: Key

Section: A/C

Read the directions carefully.

Write neatly in pencil and **show all your work**

(you will only get credit for what you put on paper).

Please do not share calculators during the test.

If you have trouble during the test, feel free to ask me for help.

Score:__________

1. Suppose that a large tank initially holds 300 gal of water where 50 lb of salt have been dissolved. Another brine solution is pumped in at a rate of 3 gal/min with a concentration of 2 lb/gal. Assume that the mixture is stirred uniformly before being pumped out at a rate of 4gal/min. Determine the IVP, **but do not solve**, for the amount of salt A(t) for time t.

$$\frac{dA}{dt} = R_i - R_o$$

$$R_i = \left(\frac{3\text{gal}}{\text{min}}\right)\left(\frac{2\text{lb}}{\text{gal}}\right) = \frac{6\text{ lb}}{\text{min}}$$

$$R_o = \left(\frac{4\text{gal}}{\text{min}}\right)\left(\frac{A(t)}{300-t}\ \frac{\text{lb}}{\text{gal}}\right) = \frac{4A(t)}{300-t}\ \frac{\text{lb}}{\text{min}}.$$

$$\begin{cases} \dfrac{dA}{dt} = 6 - \dfrac{4A}{300-t} \\ A(0) = 50 \end{cases}$$

2. Classify the following differential equations by order and linearity. If the equation is nonlinear, state why.

a. $x^2 \dfrac{d^3y}{dx^3} + \sin(x)\dfrac{dy}{dx} = y$

3rd order

Linear.

b. $\dfrac{d^3y}{dx^3} + \left(\dfrac{y}{e^x}\right)\dfrac{dy}{dx} = 4$

3rd order

Nonlinear - we have $y\dfrac{dy}{dx}$

c. $\left(\dfrac{d^2y}{dx^2}\right)^2 + \dfrac{dy}{dx} = \cos(x)$

2nd order

Nonlinear - we have $\left(\dfrac{d^2y}{dx^2}\right)^2$.

3. Determine a region in the xy-plane for $\frac{dy}{dx} = \sqrt{xy}$ so that the differential equation would have a unique solution through a point (x_0, y_0) in the region.

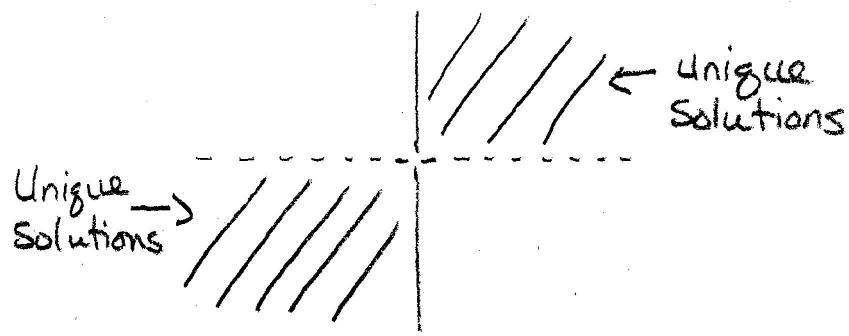
$\frac{dy}{dx} = \sqrt{xy} = f(x,y)$

Regions: $x \geq 0; y \geq 0.$
$x \leq 0, y \leq 0$

$\frac{\partial f(x,y)}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}} = \frac{1}{2}\sqrt{\frac{x}{y}}$

Regions: $x \geq 0; y > 0$
$x \leq 0, y < 0$

Region: $x \geq 0; y > 0.$
$x \leq 0, y < 0$



4. Solve the IVP $\frac{dy}{dx} = \frac{y^2-1}{x^2-1}$; $y(2) = 2$

$\int \frac{dy}{y^2-1} = \int \frac{dx}{x^2-1}$

$\frac{1}{y^2-1} = \frac{A}{y+1} + \frac{B}{y-1} = \frac{-\frac{1}{2}}{y+1} + \frac{\frac{1}{2}}{y-1}$

$1 = A(y-1) + B(y+1)$

$y=1 \quad 2B = 1 \Rightarrow B = \frac{1}{2}$

$y=-1 \quad -2A = 1 \Rightarrow A = -\frac{1}{2}$

$\int \left(\frac{-\frac{1}{2}}{y+1} + \frac{\frac{1}{2}}{y-1} \right) dy = \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} \right) dx$

$\frac{1}{2}(\ln|y-1| - \ln|y+1|) = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + c.$

$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$

$\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right| + c_1$

$\frac{y-1}{y+1} = c_2\left(\frac{x-1}{x+1}\right)$

$\frac{1}{3} = \frac{2-1}{2+1} = c_2\left(\frac{2-1}{2+1}\right) = \frac{c_2}{3} \Rightarrow c_2 = 1.$

$\frac{y-1}{y+1} = \frac{x-1}{x+1}$

$y-1 = (y+1)\left(\frac{x-1}{x+1}\right) = y\left(\frac{x-1}{x+1}\right) + \frac{x-1}{x+1}$

$y\left(1 - \frac{x-1}{x+1}\right) = 1 + \frac{x-1}{x+1}$

$y\left(\frac{2}{x+1}\right) = \frac{2x}{x+1}$

$y = x.$

5. For $\frac{dy}{dx} = y^2(4-y^2)$

Find the critical points and phase portrait. Classify each critical point as either asymptotically stable, unstable, semi-stable. Then by hand, sketch typical solution curves in the regions in the xy-plane determined by the graphs of the equilibrium solutions.

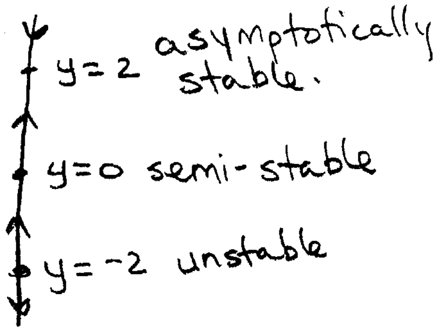
$$\frac{dy}{dx} = y^2(4-y^2) = 0$$

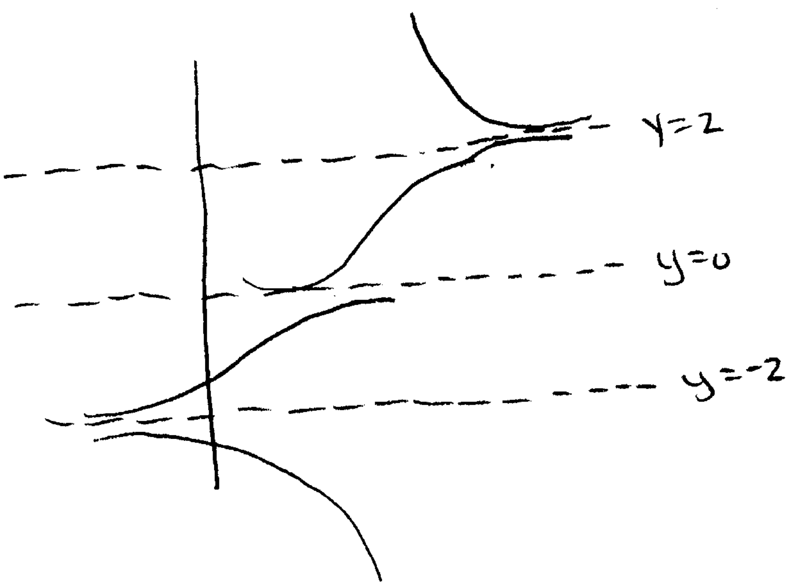
$y^2 = 0$ $\qquad$ $4 - y^2 = 0$

$y = 0$ $\qquad$ $y^2 = 4$

$y = \pm 2.$

Int	TV	+/-	↑/↓
$(-\infty,-2)$	-3	$-$	↓
$(-2,0)$	-1	$+$	↑
$(0,2)$	1	$+$	↑
$(2,\infty)$	3	$-$	↓





6. Find the general solution to $(x^2-9)\frac{dy}{dx}+4xy=6$

$$\frac{dy}{dx}+\left(\frac{4x}{x^2-9}\right)y=\frac{6}{x^2-9}$$

$$\text{IF}: e^{\int \frac{4x}{x^2-9}dx} = e^{2\int \frac{2x}{x^2-9}dx} = e^{2\ln|x^2-9|} = e^{\ln|(x^2-9)^2|} = (x^2-9)^2$$

$u = x^2-9$

$du = 2x$

$$(x^2-9)^2\left[\frac{dy}{dx}+\left(\frac{4x}{x^2-9}\right)y=\frac{6}{x^2-9}\right]$$

$$(x^2-9)^2\frac{dy}{dx}+4x(x^2-9)y=6(x^2-9)$$

$$\frac{d}{dx}\left[(x^2-9)^2y\right]=6(x^2-9)$$

$$\int\frac{d}{dx}\left[(x^2-9)^2y\right]dx=6\int(x^2-9)dx$$

$$(x^2-9)^2y=6\left(\frac{x^3}{3}-9x\right)+c=2x^3-54x+c$$

So $y(x)=\dfrac{2x^3-54x+C}{(x^2-9)^2}$